Complexity and Exactness in Polynomial Optimization

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Let us dispense with mathematical details

Theorem. Consider a **feasible** system of the form

$$\boldsymbol{f}(\boldsymbol{x}) \leq 0, \quad \boldsymbol{Ax} \leq \boldsymbol{b}$$

- $\boldsymbol{f}(\boldsymbol{x}) : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ is of degree 3, with integer coefficients
- A and b have integer coefficents; $Ax \leq b$ defines a polytope in $\mathbb{R}^n$.

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- $\boldsymbol{f(x)}$ ***linear*** $\to$ classical theory of polyhedra yields rational solution of polynomial size
- $\boldsymbol{f(x)}$ ***quadratic*** $\to$ Vavasis $\sim$ 1990 does same

Same proof

Theorem. Consider a **feasible** system of the form

$$\begin{aligned} f_i(x) &\leq 0, \quad 1 \leq i \leq m \\ Ax &\leq b \end{aligned}$$

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$f(x)$ quadratic $\rightarrow$ Vavasis

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It is strongly NP-hard to test if there is a **rational** feasible solution of **polynomial size** (polynomially many bits).

More

Theorem. Consider an optimization problem of the form

$$F^* \doteq \min \quad F(x)$$
$$\text{s.t.} \quad Ax \le b$$

- Here, $\boldsymbol{F(x)}$ is ***cubic***
- It is known that $\boldsymbol{F^*} = -\infty$.
- F, A and b have integer coefficients

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Theorem. Given $\boldsymbol{Q(x)}$ ***quadratic***, A, b with integer coefficients, it is strongly NP-hard to test whether

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Murty and Kabady, 1987

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However: if the problem is unbounded, there is a rational feasible ray along which $Q(x) \to -\infty$.

In fact unboundedness checking is in NP.

More background

Theorem. Given $\boldsymbol{T}(\boldsymbol{x})$ ***cubic***, A, b with integer coefficients, **if**

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Such a rational ray **always exists** over $\mathbb{R}^n$.

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- It is the simplest one I have. Worse behaviors can be produced.

An unlucky QCQP

```
  Minimize
     x2
  Subject To
  o1: -2 x1 + [ x1^2 + x2^2 - sneaky^2 ] >= 2
  o2:  2 x1 + [ x1^2 + x2^2 ] >= 2
  e1: [ .1 x1^2 + x2^2 ] <= 2
  bad:  distraction + [ sneaky^2 ] >= 0.1
joke1: - a + [ distraction^2 ] <= 0.0
joke2: - b + [ a^2 ] <= 0.0
cruel:  - sneaky + [ b^2 ]  <= 0.0

Bounds
x1 free
x2 free
End
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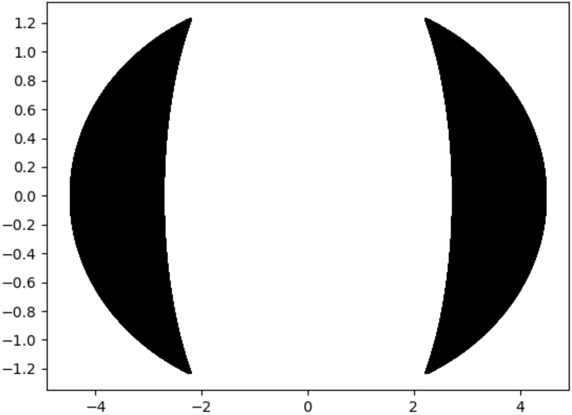
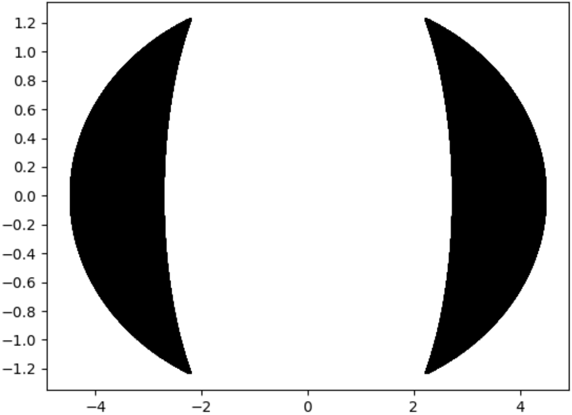
Gurobi 9, SCIP, etc: value ≈ -1.4142 Wrong, actual value ≈ -1.22

Projection of the feasible region to (x_1, x_2) plane



Unique optimal solution,

Projection of the feasible region to (x_1, x_2) plane



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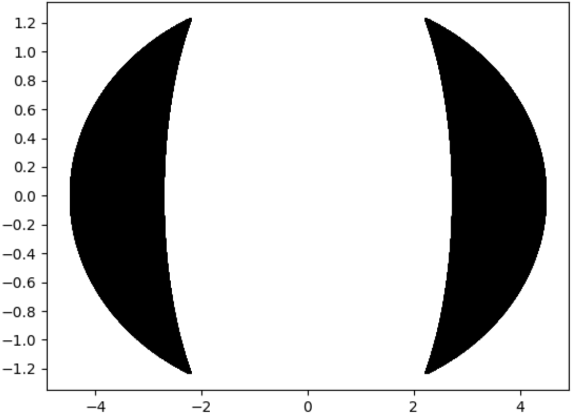

Unique optimal solution, which changes smoothly with small changes of coefficients: problem is "well-posed"

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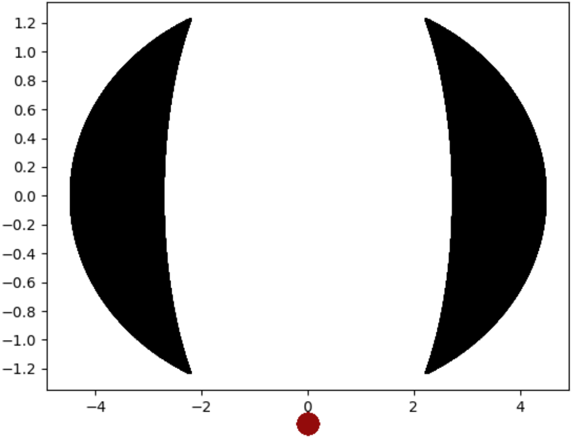
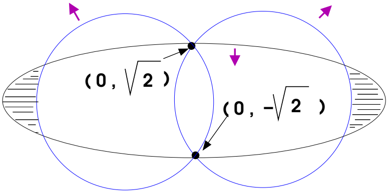


Solvers produce a point far from the feasible region

What's going on? Here is the problem that solvers think they see

$$\begin{aligned} \min \quad & x_2 \\ -2x_1 + x_1^2 + x_2^2 &\geq 2 \\ 2x_1 + x_1^2 + x_2^2 &\geq 2 \\ \frac{x_1^2}{10} + x_2^2 &\leq 2 \end{aligned}$$



But actually it has to handle

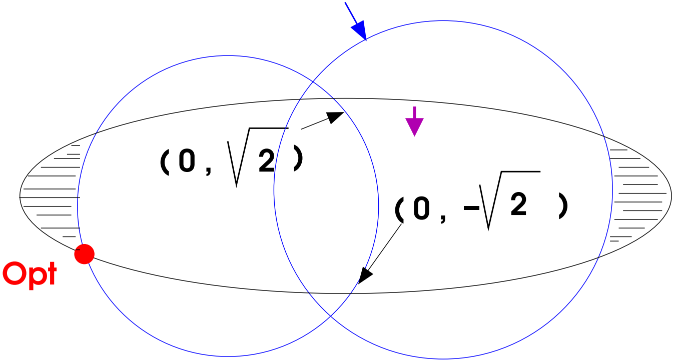
$$\begin{array}{rcl} & \min \quad x_2 & \\ -2x_1 + x_1^2 + x_2^2 & \geq & 2 + \text{sneaky}^2 \\ 2x_1 + x_1^2 + x_2^2 & \geq & 2 \\ \frac{x_1^2}{10} + x_2^2 & \leq & 2 \end{array}$$

and

$$\begin{array}{rcl} \text{distraction} + \text{sneaky}^2 & >= & 1/10 \\ -\text{a} + \text{distraction}^2 & \leq & 0 \\ -\text{b} + \text{a}^2 & \leq & 0 \\ -\text{sneaky} + \text{b}^2 & \leq & 0 \end{array}$$

Second system implies sneaky > 0

The true feasible region in (x_1, x_2) projection



Ball on right $-2\boldsymbol{x}_1 + \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 \geq 2 + \text{sneaky}^2$

Some potential reactions

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- This is just roundoff error. We are used to roundoff error, nothing new here.
- The infeasible solution with $x_2 = -\sqrt{2}$ is feasible 'to machine tolerance'. Nothing to worry about.

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- The real problem is that a convex relaxation could cut-off the infeasible solution. The relaxation could in fact be provided by the same code that produced the infeasible solution, as an option.

Relaxations applied to difficult example

The system on two real variables, $\boldsymbol{d}$ and $\boldsymbol{s}$,

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Also (SDP) $w_{d,d} \geq d^2$ and we 'remember' that $\boldsymbol{w_{s,s}}$ is a stand-in for $\boldsymbol{s}^2$ to 'get':

$$\boldsymbol{s}^2 + \boldsymbol{s} \geq 1 + \boldsymbol{d}^2 - \boldsymbol{d}$$

which is **nonconvex**! But, from Domes & Neumaier (2010), implemented in SCIP and ANTIGONE:

$$\boldsymbol{s}^2 + \boldsymbol{s} \geq 3/4$$

This implies $\boldsymbol{s} \geq 0.323$.

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Using SDP, the RHS is always at least $1 + \boldsymbol{d}^2 - \boldsymbol{d}$. So $\boldsymbol{w_{s,s}} + \boldsymbol{s} \geq 1 + \boldsymbol{d}^2 - \boldsymbol{d}$

So (two blue inequalities) $11\,\boldsymbol{s} \geq 3/4$

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$$d_1 + d_N = \frac{1}{2}, \quad 0 \leq d_1, \quad d_i^2 \leq d_{i+1} \; (1 \leq i \leq N-1)$$

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Lemma. Unless $\epsilon < 2^{-2^N}$ cannot cut-off solution with $\boldsymbol{d_N} = 0$

Another tough example

```
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e1: [ .1 x1^2 + x2_1^2 + x2_2^2 - x4_1^2 - x4_2^2 ] <= 0



op1: -2 x3 + [ x3^2 + x4_1^2 + x4_2^2 - sneaky^2 ] >= 2
op2:  2 x3 + [ x3^2 + x4_1^2 + x4_2^2 ] >= 2
ep1: [ .1 x3^2 + x4_1^2 + x4_2^2 ] <= 2

  bad:  distraction + [ sneaky^2 ]         >= 0.1
joke1: - a + [ distraction^2 ] <= 0.0
joke2: - b + [ a^2 ]            <= 0.0
cruel:  - sneaky + [ b^2 ]       <= 0.0



End
```

Relevant work

- **Renegar** – 80s, 90s. Computing solutions to algebraic systems.
- **Basu, Pollack, Roy** (2006). Algorithms in Real Algebraic Geometry.
- **Geronimo, Perrucci, Tsigaridas** (2013). Minima of polynomials over semi-algebraic sets.
- **O'Donnell** (2017). SOS systems probably require exponentially many bits.
- **Waki, Nakata, Muramatsu** (2012) SDPs arising in SOS systems can give rise to premature 'proofs'.

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Exploring the Power Flow Solution Space Boundary

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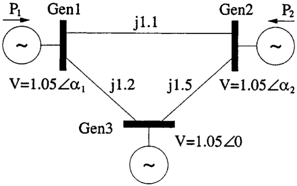

Fig. 6. Three bus system.

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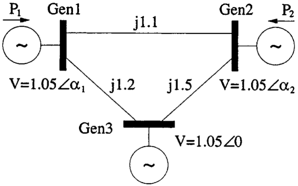

Fig. 6. Three bus system.

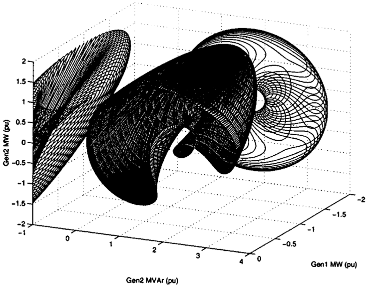

Fig. 13. Solution space, P_1–Q_2–P_2 view.

ACOPF problem

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & g_i(x) \le 0, & 1 \le i \le m \end{array}$$

$\boldsymbol{g_i(x)}$ nonlinear, nonconvex

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GO competition, the practice:

$$\begin{aligned} \min \quad f(x) + \sum_i \Phi_i(\sigma_i) & \\ \text{s.t.} \quad g_i(x) & \leq \sigma_i, \quad 1 \leq i \leq m \\ \sigma & \geq 0 \end{aligned}$$

$\Phi_{\boldsymbol{i}}()$: convex ***penalty***

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