Linear bilevel optimitization: overview and recent results

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Outline

- Bilevel optimisation
- Linear Bilevel problems

Bilevel optimization: what is it and a bit of history

Bilevel Optimization Problem

$$
\begin{aligned}
\max_{x,y} \quad & f(x,y) \\
\text{s.t.} \quad & (x,y) \in X \\
& y \in S(x) \\
\text{where} \quad & S(x) = \operatorname*{argmax}_{y} g(x,y) \\
& \text{s.t.} (x,y) \in Y
\end{aligned}
$$

Adequate framework for Stackelberg game

- Leader: 1st level,
- Follower: 2nd level,
- Leader takes follower's optimal reaction into account.

Heinrich von Stackelberg
(1905 - 1946)

First OR paper on bilevel optimization

Bracken & McGill (Op.Res.1973): First bilevel model, structural properties, military application.

Mathematical Programs with Optimization Problems in the Constraints

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

(Received October 5, 1971)

This paper considers a class of optimization problems characterized by constraints that themselves contain optimization problems. The problems in the constraints can be linear programs, nonlinear programs, or two-sided optimization problems, including certain types of games. The paper presents theory dealing primarily with properties of the relevant functions that result in convex programming problems, and discusses interpretations of this theory. It gives an application with linear programs in the constraints, and discusses computational methods for solving the problems.

Applications

- Economic game theory
- Production planning
- Revenue management
- Security
- …

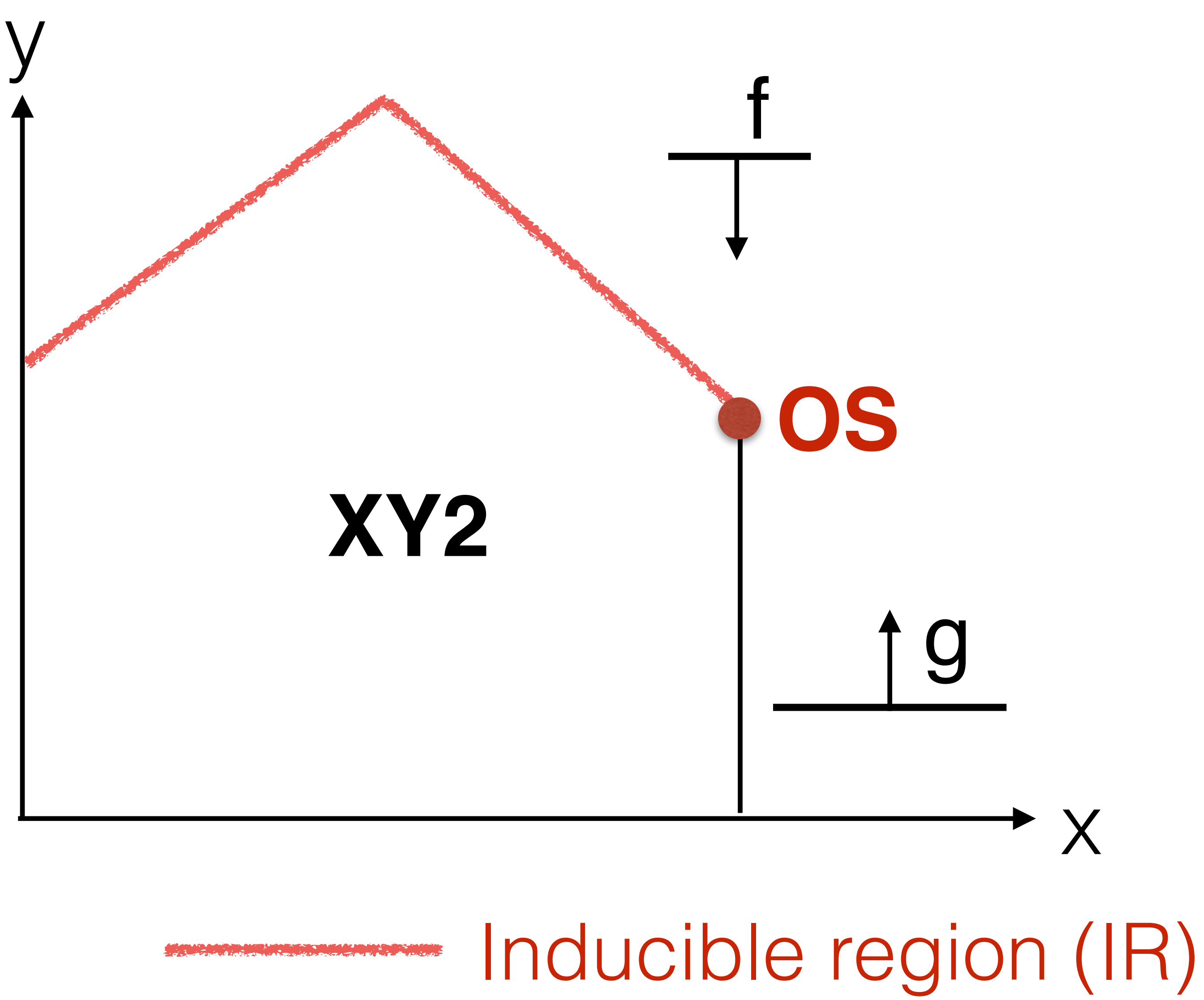
Linear bilevel problems:

Linear BP: maybe the simplest ones

$$\begin{aligned} \min_{x,y} \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & \min_{y} \quad fy \\ & \text{s.t.} \quad Cx + Dy \geq b \end{aligned}$$

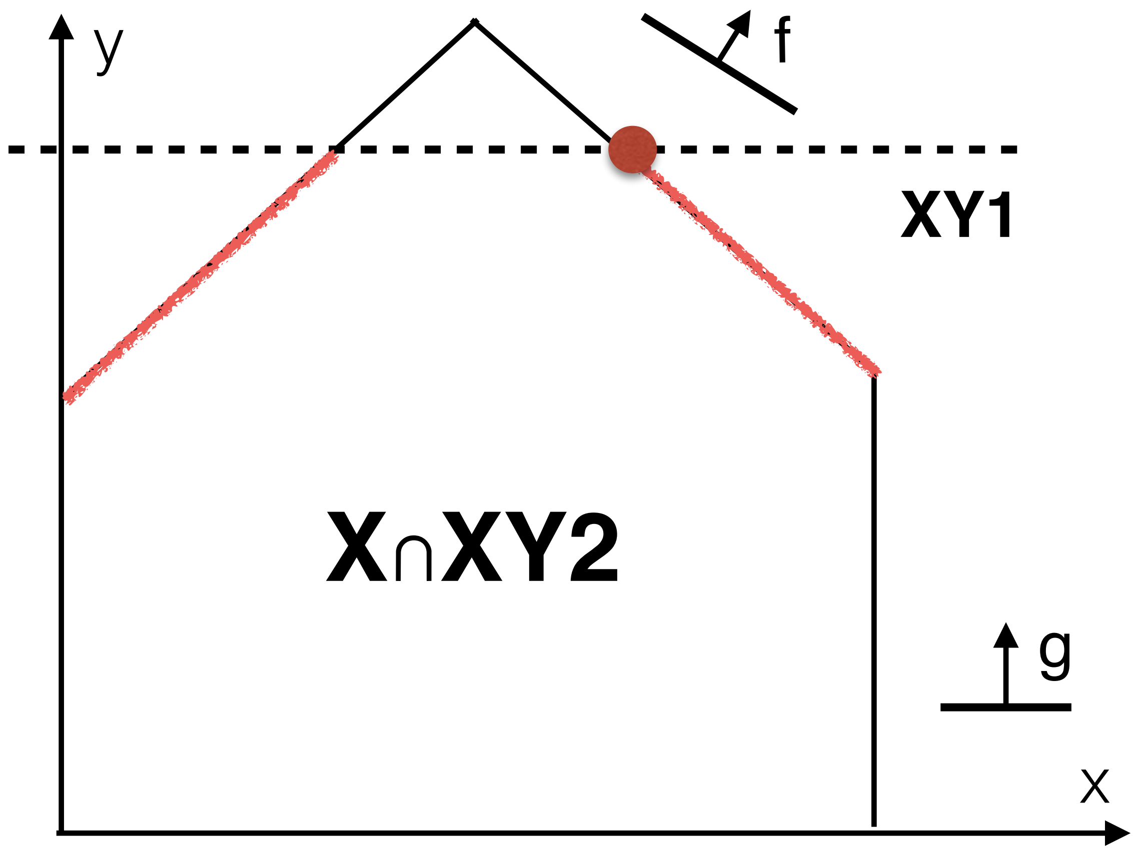
Example of linear BP

$$\max_{x,y} \quad f_1 x + f_2 y$$
$$\text{s.t.} \quad \max_y g_1 x + g_2 y$$
$$\text{s.t.} (x, y) \in XY2$$



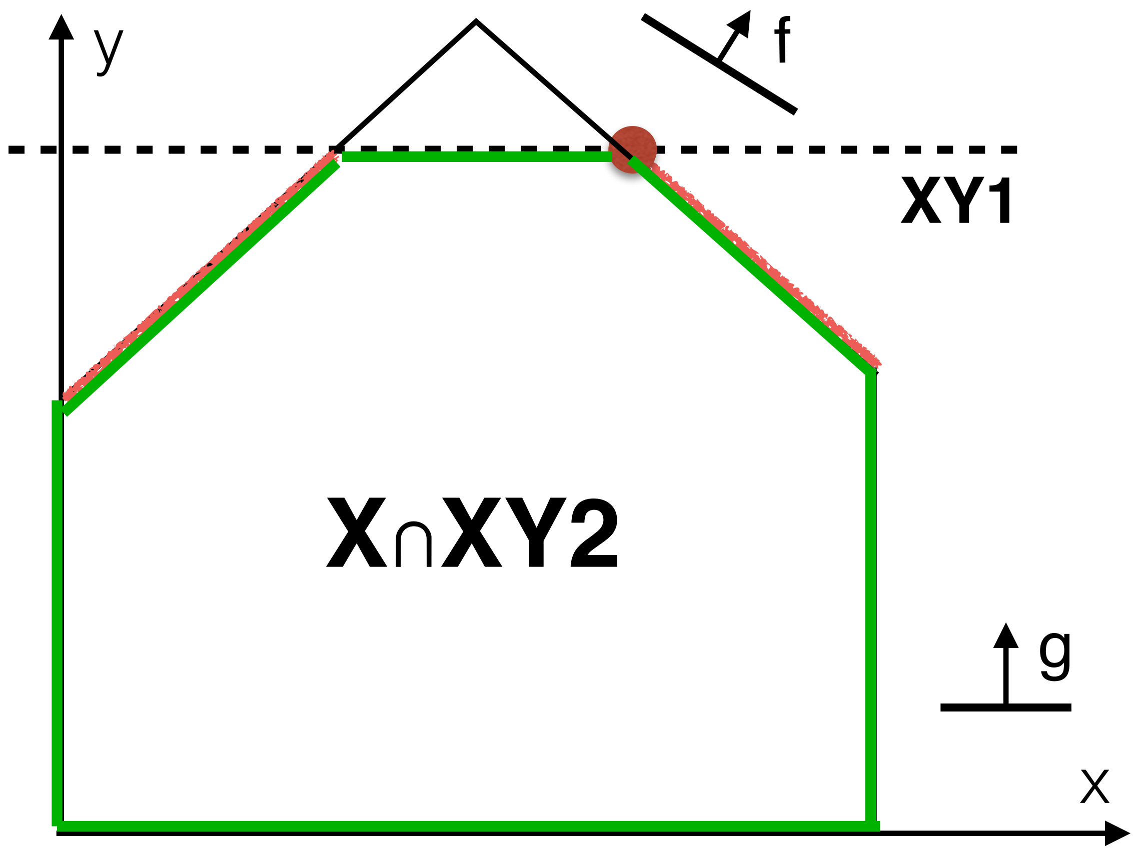
Coupling constraints

The follower sees only the second level constraints



$$\max_{x,y} \quad f_1 x + f_2 y$$
$$\text{s.t.} \quad x \in X$$
$$(x, y) \in XY1$$
$$\max_{y} g_1 x + g_2 y$$
$$\text{s.t.} (x, y) \in XY2$$

High point relaxation

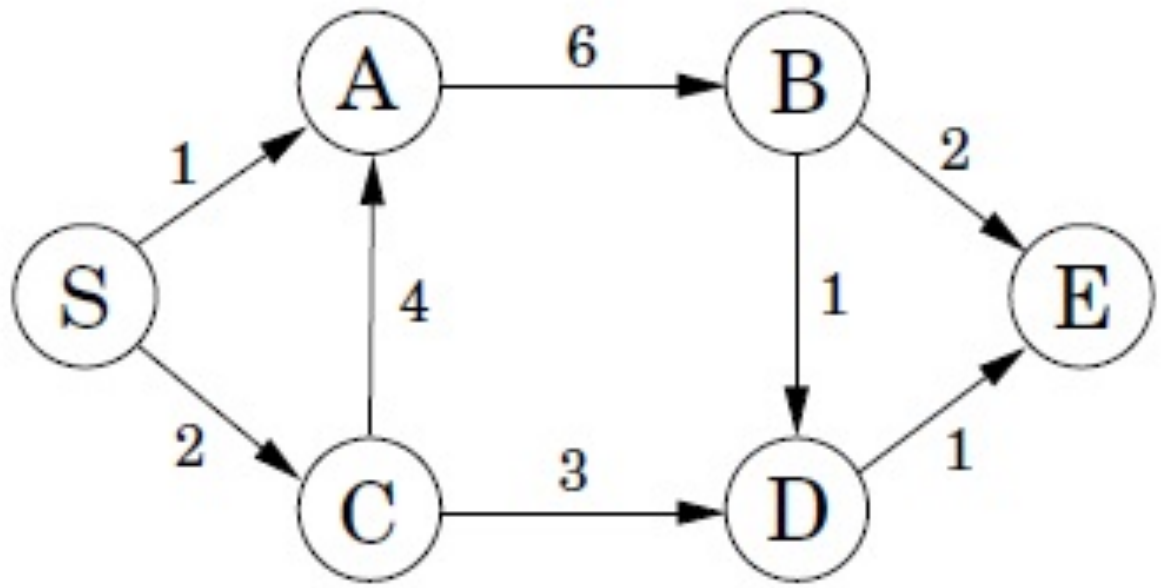


$$\begin{aligned} \max_{x,y} \quad & f_1 x + f_2 y \\ \text{s.t.} \quad & x \in X \\ & (x, y) \in XY1 \\ & \max_{y} g_1 x + g_2 y \\ & \text{s.t.} (x, y) \in XY2 \end{aligned}$$

$$X \cap XY1 \cap XY2 = \text{High Point Relaxation (HPR)}$$

Understanding the model: example

The shortest path interdiction problem



Interdict set S of at most K arcs in order to maximize length of shortest path from s to t in (V, A/S)

- x_{ij} = 1 if (i,j) is interdicted
- y_{ij} = 1 if (i,j) belongs to (shortest) path
- $Ny = \boldsymbol{\delta}$

The shortest path interdiction problem

$$
\begin{aligned}
\max_{x,y} \quad & cy \\
\text{s.t.} \quad & ex \leq K \\
& x \in \{0,1\}^m \\
& \min_{y} \quad cy \\
& \text{s.t.} \quad Ny = \delta \\
& \qquad\quad x + y \leq e \\
& \qquad\quad y \geq 0
\end{aligned}
$$

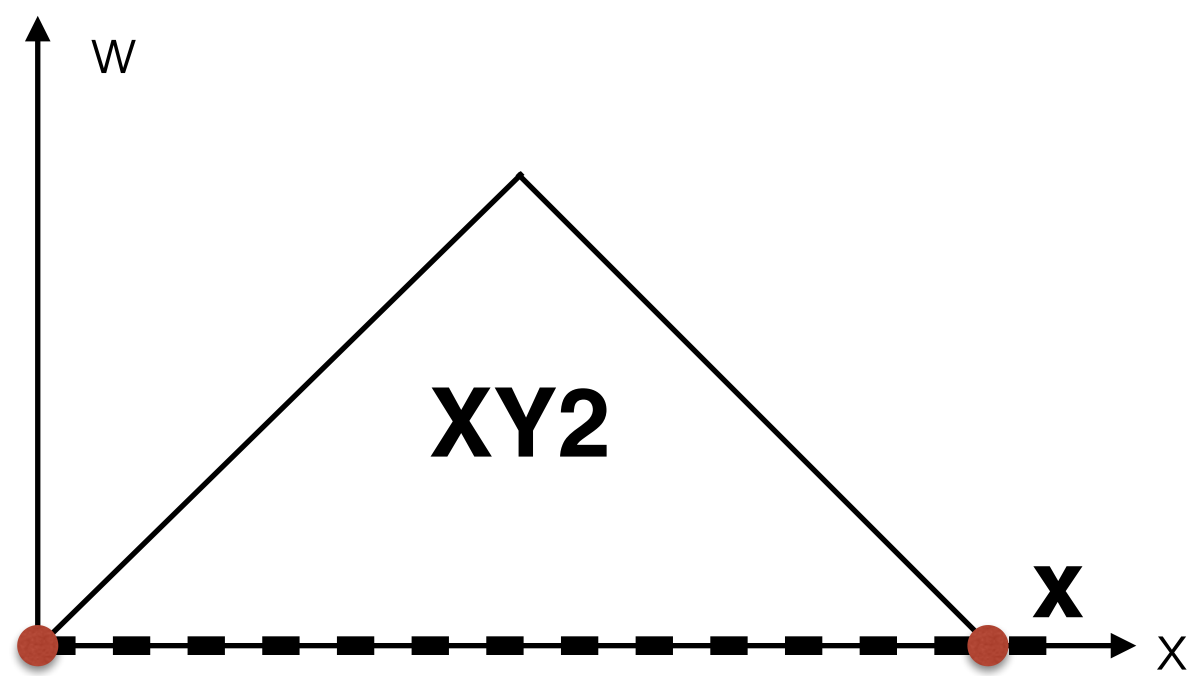
The shortest path interdiction problem

$$
\begin{aligned}
\max_{x,y} \quad & cy \\
\text{s.t.} \quad & ex \leq K \\
& x \in \{0,1\}^m \\
& \min_{y} \quad cy \\
& \text{s.t.} \quad Ny = \delta \\
& \qquad\quad x + y \leq e \\
& \qquad\quad y \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max_{x,y} \quad & (c + Mx)y \\
\text{s.t.} \quad & ex \leq K \\
& x \in \{0,1\}^m \\
& \min_{y} \quad (c + Mx)y \\
& \text{s.t.} \quad Ny = \delta \\
& \qquad\quad y \geq 0
\end{aligned}
$$

0/1 Programming is a special case of BLO
(Audet et al. 1997)

$$x \in \{0,1\} \Leftrightarrow v = 0 \text{ and } v = \underset{w}{\text{argmax}}\{w : w \leq x, w \leq 1 - x, w \geq 0\}$$



0/1 Programming is a special case of BLO
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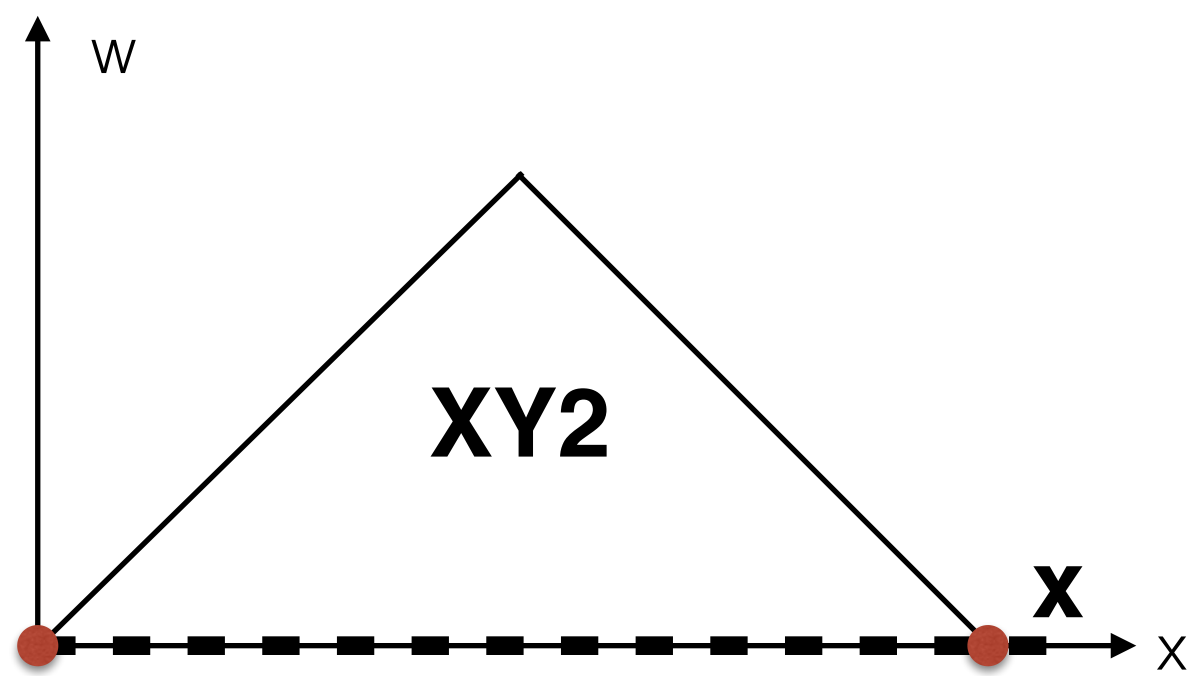
$$x \in \{0,1\} \Leftrightarrow \boxed{v = 0} \text{ and } v = \underset{w}{\operatorname{argmax}}\{w : w \leq x, w \leq 1-x, w \geq 0\}$$

Coupling constraint



Linear BP

$$\begin{aligned} \min_{x,y} \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & \min_{y} \quad fy \\ & \text{s.t.} \quad Cx + Dy \geq b \end{aligned}$$

Linear BP

- Linear BP is strongly NP-hard (Hansen et al. 1992): Kernel reduces to linear BP.

- MILP is a special case of Linear BP

Linear BP

(Bialas & Karwan(1982), Bard(1983)).

- IR is the union of faces of HPR
- IR is connected if there is no coupling constraint
- If Linear BP is feasible, then there exists an optimal solution which is a vertex of HPR.

→ K-th best algorithm

Linear BP- single level reformulation

$$\begin{aligned} \min_x \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & \min_y fy, \\ & \text{s.t.} \quad Dy \geq b - Cx \quad (\lambda) \end{aligned}$$

$$\begin{aligned} \min_x \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & Dy \geq b - Cx \\ & \lambda D = f \\ & \boxed{\lambda(Dy - b + Cx) = 0} \\ & \lambda \geq 0 \end{aligned}$$

- Branch & Bound (Hansen et al.1992)
- Branch & Cut (Audet et al. 2007)

Linear BP

$$
\begin{aligned}
\min_x \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Dy \geq b - Cx \\
& \lambda D = f \\
& \lambda(Dy - b + Cx) = 0 \\
& \lambda \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_x \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Cx + Dy \geq b \\
& \lambda D = f \\
& \lambda \geq 0 \\
& \lambda \leq M_d z \\
& Cx + Dy \leq b + M_p(1 - z) \\
& z \in \{0, 1\}^m
\end{aligned}
$$

Fortuny-Amat, McCarl (1981)

Linear BP

$$
\begin{aligned}
\min_{x} \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Cx + Dy \geq b \\
& \lambda D = f \\
& \lambda \geq 0 \\
& \lambda \leq M_d z \\
& Cx + Dy \leq b + \boxed{M_p}(1 - z) \\
& z \in \{0, 1\}^m
\end{aligned}
$$

Often available

Linear BP: valid big M

Trial-and-error tuning procedure:

- Choose some arbitrary values for M_p and M_d
- Solve the MIP
- If some M_p or M_d appear in “active” constraints, increase them and iterate.

Linear BP: valid big M

Pineda and Morales (2018): This trial-and-error method is wrong

$$\begin{aligned}
\max_{x \in \mathbb{R}} \quad & z = x + y \\
\text{s.t.} \quad & 0 \leq x \leq 2 \\
& \min_{y \in \mathbb{R}} \quad y \\
& \quad \text{s.t.} \quad y \geq 0 \quad (\lambda_1) \\
& \quad\quad\quad x - 0.01y \leq 1 \quad (\lambda_2)
\end{aligned}$$

$$\begin{aligned}
\max_{x \in \mathbb{R}, y \in \mathbb{R}} \quad & z = x + y \\
\text{s.t.} \quad & 0 \leq x \leq 2 \\
& y \geq 0 \\
& x - 0.01y \leq 1 \\
& 1 - \lambda_1 - 0.01\lambda_2 = 0 \\
& \lambda_1, \lambda_2 \geq 0 \\
& \lambda_1 \leq u_1 M_1^D \\
& y \leq (1 - u_1) M_1^P \\
& \lambda_2 \leq u_2 M_2^D \\
& -x + 0.01y + 1 \leq (1 - u_2) M_2^P \\
& u_1, u_2 \in \{0, 1\}
\end{aligned}$$

Linear BP: valid big M

$$\max_{x\in\mathbb{R},y\in\mathbb{R}} \quad z = x + y$$
$$\text{s.t.} \quad 0 \le x \le 2$$
$$y \ge 0$$
$$x - 0.01y \le 1$$
$$1 - \lambda_1 - 0.01\lambda_2 = 0$$
$$\lambda_1, \lambda_2 \ge 0$$
$$\lambda_1 \le u_1 M_1^D$$
$$y \le (1 - u_1) M_1^P$$
$$\lambda_2 \le u_2 M_2^D$$
$$-x + 0.01y + 1 \le (1 - u_2) M_2^P$$
$$u_1, u_2 \in \{0, 1\}$$

D2 is bounded: $\lambda_1 \le 1, \lambda_2 \le 100$

The trial-and-error procedure stops for

$$M_1^P = M_2^P = 200, M_1^D = M_1^D = 50$$

with a solution that is not optimal.

Linear BP: valid big M_d

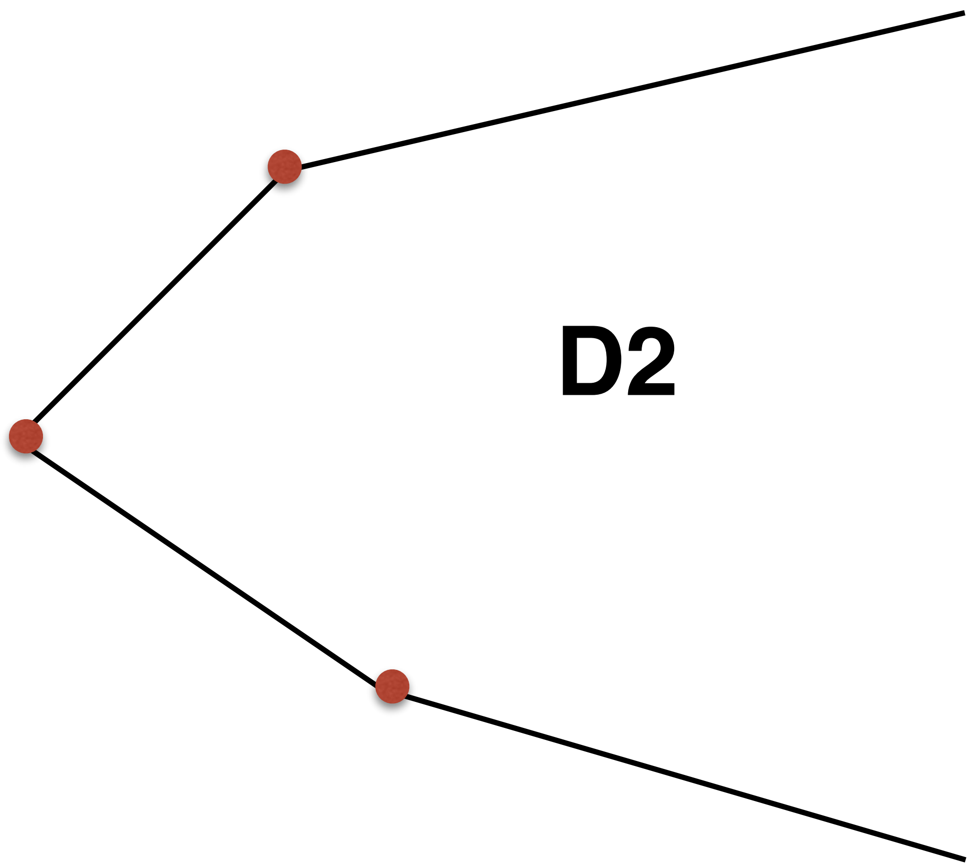
- Upper bound on any $\boldsymbol{\lambda}$ in D2= $\{\lambda \geq 0 : \lambda D = f\}$

- D2 is often unbounded

- In any feasible solution (x^*,y^*, $\boldsymbol{\lambda}^*$) to LBP,

- $\boldsymbol{\lambda}^*$ is the optimal solution of an LP on D2
- $\boldsymbol{\lambda}^*$ is a vertex of D2

Linear BP: valid big M_d



Linear BP: valid big M_d

(Kleinert, Labbé, Schmidt, Plein 2020)

Problem valid M_d: given A, b, M
Question: is $x_i \leq M$ for every vertex x of $Ax \leq b$?

Strongly co-NP complete

Its complement:
Question: does it exists a vertex x of $Ax \leq b$ such that $x_i \leq M$?

Strongly NP-complete

Linear BP: reformulation using strong duality

$$\begin{aligned} \min_x \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & \min_y fy, \\ & \text{s.t.} \quad Dy \geq b - Cx \quad (\lambda) \end{aligned}$$

$$\begin{aligned} \min_x \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & Dy \geq b - Cx \\ & \lambda D = f \\ & \lambda \geq 0 \\ & fy \leq \lambda(b - Cx) \end{aligned}$$

Linear BP: a valid Primal-Dual inequality

(Kleinert, Labbé, Schmidt, Plein)

$$\boxed{fy + \lambda Cx - \lambda b \leq 0}$$

$$C_i^- = \min_x C_{i.}x \text{ s.t. } (x, y, \lambda) \text{ is 'feasible'}$$

$\Rightarrow$ Valid linear inequality: $fy + C^- x - \lambda b \leq 0$

Linear BP: a valid Primal-Dual inequality

(Kleinert, Labbé, Schmidt, Plein)

$$fy + \lambda Cx - \lambda b \leq 0$$

McCormick (1976) envelopes: $z_i = \lambda_i (C_{i.} x)$

$$fy + \sum_i z_i - \lambda b \leq 0$$

$$z_i \geq \lambda_i^+ C_{i.} x + \lambda_i C_i^+ - \lambda_i^+ C_i^+$$

$$z_i \geq \lambda_i^- C_{i.} x + \lambda_i C_i^- - \lambda_i^- C_i^-$$

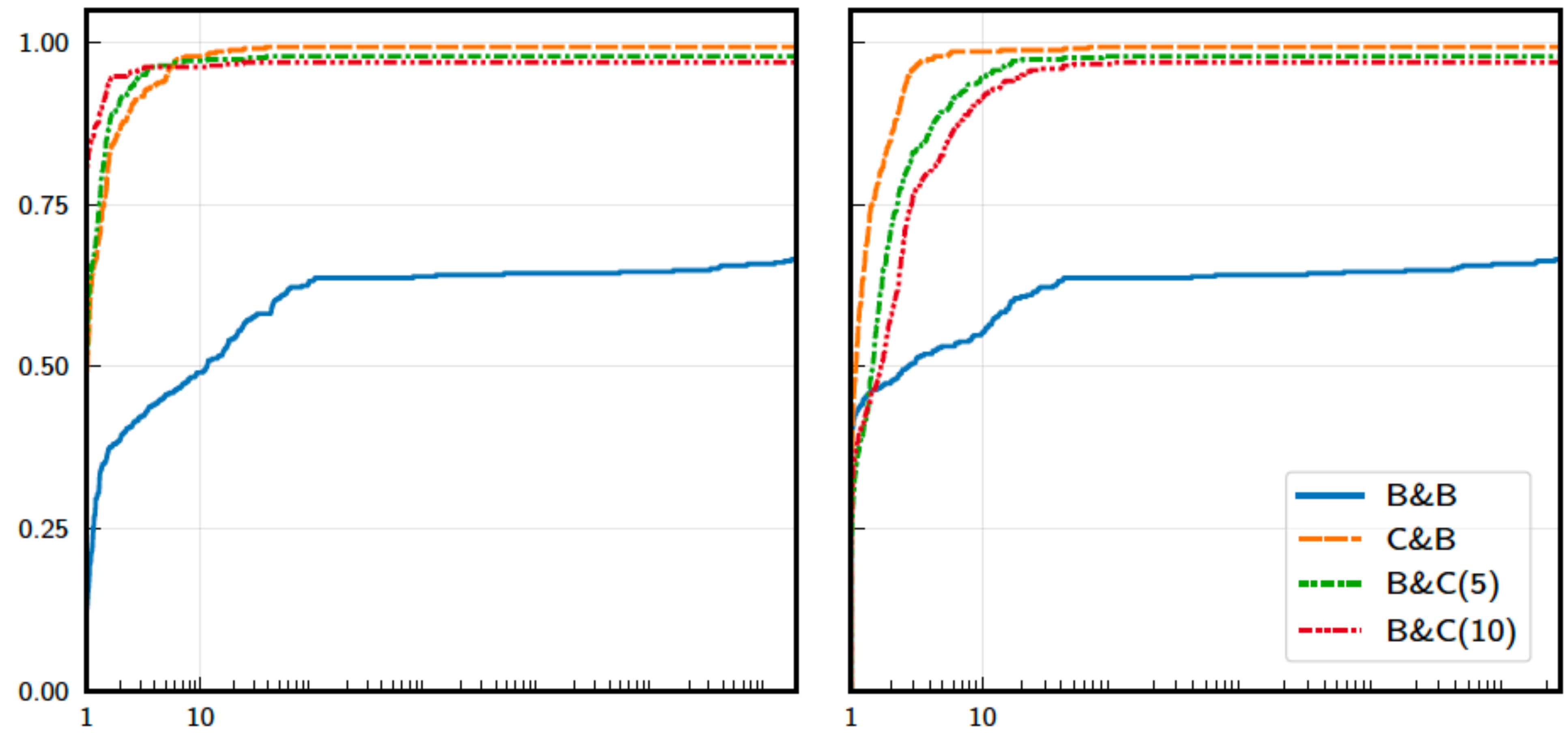


FIGURE 4. Log-scaled performance profiles for branch-and-bound nodes (left) and running times (right) over all remaining instances.

To summarize

- Model correctly your problem
- Beware of fake news
- BLP's are difficult to solve
- Wide open field of investigation