A hierarchy of port-Hamiltonian models for gas networks

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Overview

Port-Hamiltonian modeling

PH model hierarchy for gas pipes

Consistent network interconnection

Overview

Port-Hamiltonian modeling

- Energy-based paradigm.
- Stability and passivity properties.
- Consistent interconnection of diverse physical domains.
- Structure-preserving numerics.

PH model hierarchy for gas pipes

Consistent network interconnection

Overview

Port-Hamiltonian modeling

PH model hierarchy for gas pipes

- Partially-ordered (by accuracy) set of models.
- Very accurate models are complex.
- Choice of model should be application-dependent.
- Dynamical change of models if error is small.

Consistent network interconnection

Overview

Port-Hamiltonian modeling

PH model hierarchy for gas pipes

Consistent network interconnection

- Components are modeled individually.
- Energy-preserving interconnection.
- Exploiting PH structure and Kirchhoff's laws.

Port-Hamiltonian descriptor systems

Port-Hamiltonian descriptor system

A *port-Hamiltonian descriptor system* (PHDAE) is

$$\begin{aligned} E\dot{z} &= (J-R)e(z) + Bu, \\ y &= Ce(z), \end{aligned} \tag{PHDAE}$$

together with a *Hamiltonian* $\mathcal{H}(z)$, with state $z \in \mathbb{R}^n$, input and output $u, y \in \mathbb{R}^m$, effort function $e : \mathbb{R}^n \to \mathbb{R}^l$ and matrix coefficients $E \in \mathbb{R}^{l,n}$, $J, R \in \mathbb{R}^{l,l}$, $B \in \mathbb{R}^{l,m}$, $C \in \mathbb{R}^{m,l}$, such that

- $J = -J^\top$, $R = R^\top \geq 0$, and $C = B^\top$,
- $\nabla \mathcal{H}(z) = E^\top e(z)$.

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with Hamiltonian $\mathcal{H}(z)$, where $J = -J^\top$, $R = R^\top \geq 0$, $C = B^\top$.

Power balance equation and dissipation inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) = -e^\top Re + y^\top u \leq y^\top u \tag{PBE/DI}$$

$\Rightarrow$ stability and passivity properties (given e.g. $\mathcal{H} \geq 0$).

Port-Hamiltonian descriptor systems

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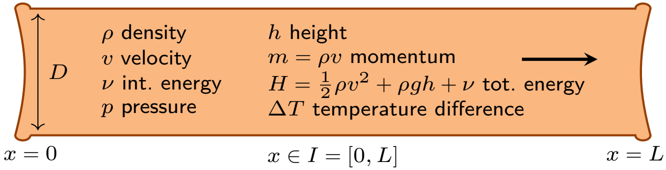
Extensions:

- State and time dependent coefficients.
- Addition of feedthrough terms.
- Generalization to PDEs.

1-D Euler equations for gas pipes (TA1)



1-D Euler equations for gas pipes

$$
\begin{aligned}
&\text{Mass} && \frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial x} = 0, \\
&\text{Momentum} && \frac{\partial m}{\partial t} + \frac{\partial}{\partial x}(mv + p) = -\lambda |v| m - \rho g \frac{\partial h}{\partial x}, \\
&\text{Energy} && \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}((H + p)v) = k \Delta T.
\end{aligned}
\tag{TA1}
$$

together with *equation of state* $p = p(\rho, \nu)$.

Energy conservation

- Total energy of the system: $\mathcal{H} = \int_0^L H \, \mathrm{d}x$.
- The third equation gives a PBE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -[v(H+p)]_0^L + k \int_0^L \Delta T \, \mathrm{d}x$$

where $[F]_0^L := F(L) - F(0)$.

- Energy enters the system in two ways:
  - boundary of the pipe ($x \in \{0, L\}$),
  - temperature difference with the pipe wall.
- Friction $\rightarrow$ internal energy: no dissipation.

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1-D Euler equations as a pHDAE

- Define state vector $z = (\rho, v, \nu)$.
- Hamiltonian $\mathcal{H}(z) = \int_0^L H(z)\, \mathrm{d}x \Rightarrow \dfrac{\delta \mathcal{H}}{\delta z}(z) = \begin{bmatrix} \frac{v^2}{2} + gh \\ \rho v \\ 1 \end{bmatrix}$.
- Rewrite the equations as:

$$E(z)\dot{z} = J(z)e(z) + B_T \Delta T,$$

1-D Euler equations as a pHDAE

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- Take a closer look to $E(z)$ and $e(z)$:

$$\overbrace{\begin{bmatrix} 1 & & \\ & \rho & \\ & & 1 \end{bmatrix}}^{E(z)} \dot{z} = J(z) \overbrace{\begin{bmatrix} \frac{v^2}{2} + gh \\ v \\ 1 \end{bmatrix}}^{e(z)} + B_T \Delta T,$$

In some sense, we have $\delta \mathcal{H} = E^\top e$.

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- Add coupled output equation:

$$E(z)\dot{z} = J(z)e(z) + \overbrace{\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}}^{B_T} \Delta T,$$

$$y_T = \underbrace{\begin{bmatrix} 0 & 0 & k \end{bmatrix}}_{C_T} e(z) = k.$$

In some sense, we have $C_T = B_T^\top$.

1-D Euler equations as a pHDAE

- Define state vector $z = (\rho, v, \nu)$.
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- Take a closer look to $J(z)$:

$$J(z) = \begin{bmatrix} & -\frac{\partial}{\partial x}(\rho\cdot) & \\ -\rho\frac{\partial}{\partial x} & & J_{23}(z) \\ & J_{32}(z) & \end{bmatrix}$$

with

$$J_{23}(z) = -\nu\frac{\partial}{\partial x} - \frac{\partial}{\partial x}(p\cdot) - \lambda|v|v,$$
$$J_{32}(z) = -\frac{\partial}{\partial x}(\nu\cdot) - p\frac{\partial}{\partial x} + \lambda|v|v.$$

Do we have $J = -J^\top$? (and in which sense?)

1-D Euler equations as a pHDAE

- In finite dimension, $J = -J^\top$ iff $e^\top J e = 0$ for all e. Do we have something similar here?
- Consider the bilinear product $\langle e, f \rangle := \int_0^L e(x)^\top f(x)\,\mathrm{d}x$. Then

$$\langle e, J(z)e \rangle = -[(\rho e_1 + (\nu + p)e_3)e_2]_0^L, \quad \forall e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}.$$

- For $e = e(z)$, this is $-[(H + p)v]_0^L$.
- If e has vanishing boundary, then $\langle e, Je \rangle = 0$.
- In general, $\langle e, J(z)e \rangle = (U(z)e)^\top Y(z)e$, where

$$U(z)e = \begin{bmatrix} \rho e_2|_{x=0} \\ -\rho e_2|_{x=L} \end{bmatrix}, \quad Y(z)e = \begin{bmatrix} e_1 + \frac{\nu+p}{\rho} e_3|_{x=0} \\ e_1 + \frac{\nu+p}{\rho} e_3|_{x=L} \end{bmatrix}$$

are *input and output boundary operators.*

1-D Euler equations as a pHDAE

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are *input and output boundary operators*.

1-D Euler equations as a pHDAE

- Add boundary control and boundary output equations:

$$\mu = U(z)e(z) = \begin{bmatrix} m|_{x=0} \\ -m|_{x=L} \end{bmatrix}, \quad \beta = Y(z)e(z) = \begin{bmatrix} \frac{H+p}{\rho}|_{x=0} \\ \frac{H+p}{\rho}|_{x=L} \end{bmatrix}.$$

- The input μ is the inward momentum, the output β is the *Bernoulli invariant*.
- We have then the descriptor system

$$\overbrace{\begin{bmatrix} E(z) \\ 0 \end{bmatrix}}^{\tilde{E}(z)} \dot{z} = \overbrace{\begin{bmatrix} J(z) \\ -U(z) \end{bmatrix}}^{\tilde{J}(z)} e(z) + \overbrace{\begin{bmatrix} B_T & \\ & I_2 \end{bmatrix}}^{\tilde{B}} \overbrace{\begin{bmatrix} \Delta T \\ \mu \end{bmatrix}}^{\tilde{u}},$$

$$\underbrace{\begin{bmatrix} y_T \\ \beta \end{bmatrix}}_{\tilde{y}} = \underbrace{\begin{bmatrix} C_T \\ Y(z) \end{bmatrix}}_{\tilde{C}(z)} e(z).$$

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1-D Euler equations as a pHDAE

Bilinear products

$$\langle e, (f, u) \rangle_{Y(z)} = \langle e, f \rangle + u^\top Y(z) e,$$
$$\langle \tilde{y}, \tilde{u} \rangle_{\mathsf{IO}} = \langle y_T, \Delta T \rangle + \beta^\top \mu.$$

Properties

(1) $\tilde{J} = -\tilde{J}^\top$: $\langle e, \tilde{J}(z) e \rangle_{Y(z)} = 0$ for all z, e;

(2) $\tilde{C} = \tilde{B}^\top$: $\langle e, \tilde{B}\tilde{u} \rangle_{Y(z)} = \langle \tilde{C} e, \tilde{u} \rangle_{\mathsf{IO}}$ for all $z, e, \tilde{u}$;

(3) $\delta\mathcal{H} = E^\top e$: $\langle \delta\mathcal{H}(z), w \rangle = \langle e(z), E(z) w \rangle_{Y(z)}$ for all z, w.

1-D Euler equations as a pHDAE

Properties

$$(1)\quad \tilde{J} = -\tilde{J}^\top \quad : \quad \langle e, \tilde{J}(z)e\rangle_{Y(z)} = 0 \text{ for all } z, e;$$

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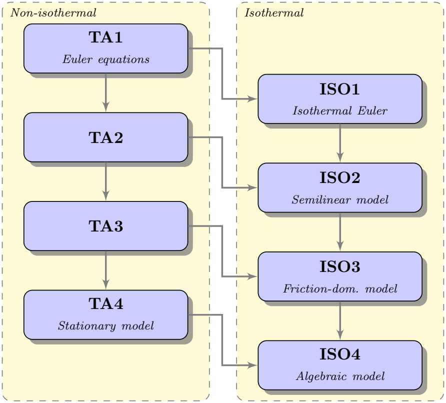
Power balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) \overset{\text{CR}}{=} \langle \delta\mathcal{H}, \dot{z}\rangle \overset{(3)}{=} \langle e, E\dot{z}\rangle_{Y(z)} \overset{\text{DAE}}{=} \langle e, \tilde{J}e + \tilde{B}\tilde{u}\rangle_{Y(z)} =$$

$$= \langle e, \tilde{J}e\rangle_{Y(z)} + \langle e, \tilde{B}\tilde{u}\rangle_{Y(z)} \overset{(1,2)}{=} \langle \tilde{C}e, \tilde{u}\rangle_{\mathsf{IO}} \overset{\text{OE}}{=} \langle \tilde{y}, \tilde{u}\rangle_{\mathsf{IO}}.$$

Model hierarchy



Model hierarchy (TRR154)[1]

- TA1 too complex for many applications.
- 8 models, 2 subhierarchies.
- Consistent PHDAE form for all but two models.
- Discard TA2 and ISO2.

[1] P. Domschke, B. Hiller, J. Lang, C. Tischendorf, Modellierung von Gasnetzwerken: Eine Übersicht. TU Darmstadt (preprint), 2017

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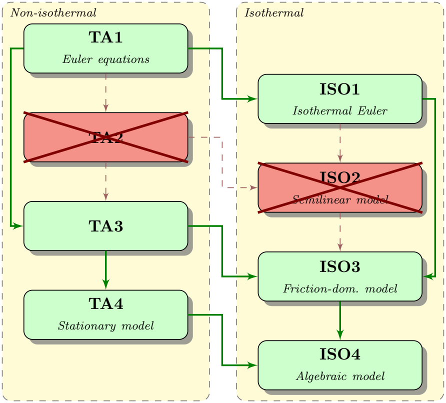

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Non-isothermal model hierarchy (TA1,TA3,TA4)

PHDAE form of TA1

$$\begin{aligned} E_{\mathsf{TA1}}\dot{z}_{\mathsf{TA}} &= J_{\mathsf{TA}}e_{\mathsf{TA1}} + B_{\mathsf{TA}}u_{\mathsf{TA}}, \\ y_{\mathsf{TA1}} &= [y_T, \beta_{\mathsf{TA1}}]^\top = C_{\mathsf{TA}}e_{\mathsf{TA1}}, \end{aligned} \qquad \text{(TA1)}$$

with $z_{\mathsf{TA}} = (\rho, v, \nu)$ and $\mathcal{H}_{\mathsf{TA1}} = \int \frac{1}{2}\rho v^2 + \rho g h + \nu \, \mathrm{d}x$.

TA3: For long time and pipes, small velocity and high friction, asymptotic analysis leads to $\mathcal{H}_{\mathsf{TA3}} = \int \rho g h + \nu \, \mathrm{d}x$, $E_{\mathsf{TA3}} = \mathrm{diag}(1, 0, 1)$, $e_{\mathsf{TA3}} = [gh, v, 1]^\top$.

TA4: Stationary assumption leads to $E_{\mathsf{TA4}} = 0$ and $\mathcal{H}_{\mathsf{TA4}} = 0$, leaving $e_{\mathsf{TA4}} = e_{\mathsf{TA3}} = [gh, v, 1]^\top$.

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Isothermal Euler equations (ISO1)

Isothermal assumption ($T \equiv T_0$ constant):

- Equation of state $p = p(\rho, T_0) = p(\rho)$.
- Restrict to $z_{\mathsf{ISO}} = (\rho, v)$ and first two equations of (TA1).
- No internal energy $\Rightarrow$ friction should be dissipation.
- Pressure potential $F(\rho)$ satisfying $\rho F'(\rho) - F(p) = p$,
- Replace ν in Hamiltonian: $\mathcal{H}_{\mathsf{ISO1}} = \int \frac{1}{2}\rho v^2 + \rho g h + F(\rho)\,\mathrm{d}x$.
- Complete with $\mu = U(z)e(z)$ and $\beta_{\mathsf{ISO1}} = Y(z)e(z)$, where $U(z)e = \rho e_2$ and $Y(z)e = e_1$. This is again a pHDAE.

$$\underbrace{\begin{bmatrix} 1 & \\ & \rho \end{bmatrix}}_{E(z_{\mathsf{ISO}})} \dot{z}_{\mathsf{ISO}} = \underbrace{\begin{bmatrix} & -\frac{\partial}{\partial x}(\rho\cdot) \\ -\rho\frac{\partial}{\partial x} & \end{bmatrix}}_{J(z_{\mathsf{ISO}})} \begin{bmatrix} \frac{v^2}{2} + gh \\ v \end{bmatrix} - \begin{bmatrix} 0 \\ \lambda\rho v|v| + \frac{\partial p}{\partial x} \end{bmatrix}$$

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Isothermal model-hierarchy (ISO1,ISO3,ISO4)

PHDAE form of ISO1

$$E_{\mathsf{ISO1}}\dot{z}_{\mathsf{ISO}} = (J_{\mathsf{ISO}} - R_{\mathsf{ISO}})e_{\mathsf{ISO1}} + B_{\mathsf{ISO}}\mu, \qquad \text{(ISO1)}$$
$$\beta_{\mathsf{ISO1}} = C_{\mathsf{ISO}}e_{\mathsf{ISO1}},$$

with $\mathcal{H}_{\mathsf{ISO1}} = \int \frac{1}{2}\rho v^2 + \rho g h + F(\rho)\,\mathrm{d}x$.

ISO3: For long time and pipes, small velocity and high friction, asymptotic analysis[2] leads to $\mathcal{H}_{\mathsf{ISO3}} = \int \rho g h + F(\rho)\,\mathrm{d}x$, $E_{\mathsf{ISO3}} = \mathrm{diag}(1,0)$, $e_{\mathsf{ISO3}} = [gh + F(\rho), v]^\top$.

ISO4: Stationary assumption leads to $E_{\mathsf{ISO4}} = 0$ and $\mathcal{H}_{\mathsf{ISO4}} = 0$, leaving $e_{\mathsf{ISO4}} = e_{\mathsf{ISO3}} = [gh + F(\rho), v]^\top$.

[2] H. Egger, J. Giesselmann, Stability and asymptotic analysis for instationary gas transport via relative energy estimates. ArXiv preprint, 2020.

Summary

Generic PHDAE pipe model

$$E(z)\dot{z} = (J(z) - R(z))e(z) + \begin{bmatrix} B_\mu & B_T \end{bmatrix} \begin{bmatrix} \mu \\ \Delta T \end{bmatrix},$$
$$\begin{bmatrix} \beta \\ y_T \end{bmatrix} = \begin{bmatrix} B_\mu^\top \\ B_T^\top \end{bmatrix} e(z),$$

- Input μ is the inward boundary momentum.
- Output β is the boundary Bernoulli invariant.
- In TA models, $R = 0$.
- In ISO models, ΔT, y_T and B_T are empty.

Network representation

A *gas network* is a oriented graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where:

- the vertices $v_1, \dots, v_n \in \mathcal{V}$ are junctions, sources and sinks.
- the edges $e^1, \dots, e^m \in \mathcal{E}$ are network components.
- for this talk: all edges are gas pipes.

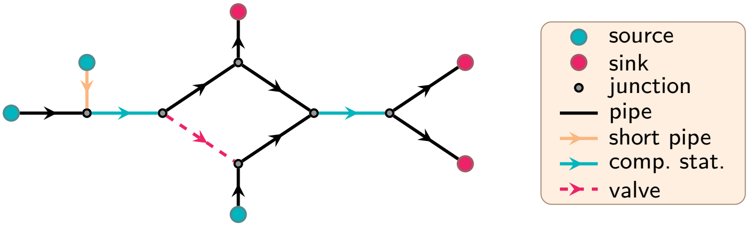

Figure: GasLib-11: a simple example with 12 vertices and 12 edges. There are 3 sources, 3 sinks, 2 compressor stations and 1 valve.

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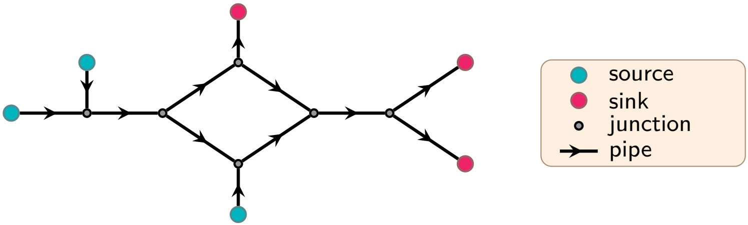

Figure: GasLib-11: a simple example with 12 vertices and 12 edges. There are 3 sources and 3 sinks.

Total system

Aggregating all gas pipe pHDAEs in block diagonal form, we get

Total network pHDAE

$$\begin{aligned} E\dot{z} &= (J-R)e(z) + B_\mu \mu + B_T \Delta T, \\ \beta &= B_\mu^\top e(z), \\ y_T &= B_T^\top e(z), \end{aligned}$$

with total Hamiltonian $\mathcal{H}(z) = \mathcal{H}_1(z^1) + \ldots + \mathcal{H}_m(z^m)$.

- All pipes are still independent.
- Coupling conditions to reduce degrees of freedom.

Total system

Aggregating all gas pipe pHDAEs in block diagonal form, we get

Total network pHDAE

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$$\beta = B_\mu^\top e(z),$$
$$y_T = B_T^\top e(z),$$

with total Hamiltonian $\mathcal{H}(z) = \mathcal{H}_1(z^1) + ... + \mathcal{H}_m(z^m)$.

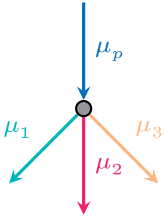
- All pipes are still independent.
- Coupling conditions to reduce degrees of freedom.

First Kirchhoff's law

The algebraic sum of momenta entering or leaving a node is 0.

Second Kirchhoff's law

The Bernoulli invariant at every node is continuously-defined.



$$\mu_1 + \mu_2 + \mu_3 = \mu_p,$$

$$\mu_p \begin{cases} \geq 0 \text{ in sources,} \\ \leq 0 \text{ in sinks,} \\ 0 \text{ in junctions.} \end{cases}$$

Conservation of total mass.

$$\beta_1^1 = \beta_0^2 = \beta_0^3 = \lambda_i$$

λ_i Bernoulli invariant in v_i.

Conservation of total energy.

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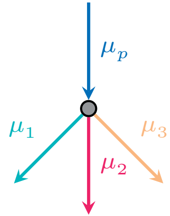

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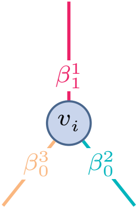

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Special incidence matrix

Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n,m}$ be the oriented incidence matrix.

Let $\mathcal{A}_s = [\mathcal{A}_0, \mathcal{A}_1] \in \mathbb{R}^{n,2m}$ be the special incidence matrix.

Let $\mathcal{A}_p = [a^p_{ij}] \in \mathbb{R}^{n,p}$ be the port incidence matrix.

Special incidence matrix

Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n,m}$ be the oriented incidence matrix, i.e.,

$$a_{ij} = \begin{cases} -1 & \text{if } e_j \text{ leaves vertex } v_i, \\ +1 & \text{if } e_j \text{ enters vertex } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Rows correspond to nodes, columns to edges.

Let $\mathcal{A}_s = [\mathcal{A}_0, \mathcal{A}_1] \in \mathbb{R}^{n,2m}$ be the special incidence matrix.

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Special incidence matrix

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Let $\mathcal{A}_s = [\mathcal{A}_0, \mathcal{A}_1] \in \mathbb{R}^{n,2m}$ be the special incidence matrix, where $\mathcal{A}_0 = [\mathcal{A} < 0]$ and $\mathcal{A}_1 = [\mathcal{A} > 0]$.

Rows correspond to nodes, columns to endpoints of edges.

Let $\mathcal{A}_p = [a^p_{ij}] \in \mathbb{R}^{n,p}$ be the port incidence matrix.

Special incidence matrix

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Let $\mathcal{A}_p = [a^p_{ij}] \in \mathbb{R}^{n,p}$ be the port incidence matrix, defined as

$$a^p_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the } j\text{-th source/sink node,} \\ 0 & \text{otherwise.} \end{cases}$$

Rows correspond to nodes, columns to sources and sinks.

Special incidence matrix

Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n,m}$ be the oriented incidence matrix.

Let $\mathcal{A}_s = [\mathcal{A}_0, \mathcal{A}_1] \in \mathbb{R}^{n,2m}$ be the special incidence matrix.

Let $\mathcal{A}_p = [a^p_{ij}] \in \mathbb{R}^{n,p}$ be the port incidence matrix.

1st Kirchhoff's law

$$\mathcal{A}_s \mu = \mathcal{A}_p \mu_p$$

2nd Kirchhoff's law

$$\mathcal{A}_s^\top \beta = \lambda$$

Network interconnection

By adding $\mathcal{A}_s \mu = 0$ and replacing $\beta = \mathcal{A}_s^\top \lambda$, we get

Interconnected network pHDAE

$$\begin{bmatrix} E & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{\mu} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} J - R & B_\mu & 0 \\ -B_\mu^\top & 0 & \mathcal{A}_s^\top \\ 0 & -\mathcal{A}_s & 0 \end{bmatrix} \begin{bmatrix} e(z) \\ \mu \\ \lambda \end{bmatrix} + \begin{bmatrix} B_T & 0 \\ 0 & 0 \\ 0 & \mathcal{A}_p \end{bmatrix} \begin{bmatrix} \Delta T \\ \mu_p \end{bmatrix},$$

$$\begin{bmatrix} y_T \\ \beta_p \end{bmatrix} = \begin{bmatrix} B_T^\top & 0 & 0 \\ 0 & 0 & \mathcal{A}_p^\top \end{bmatrix} \begin{bmatrix} e(z) \\ \mu \\ \lambda \end{bmatrix},$$

with the same total Hamiltonian $\mathcal{H}(z)$.

Additional comments

- The network pHDAE can be modified to control β instead of μ in any subset of source/sink nodes.
- In principle, different pipe models can coexist in a network:
  - special care when coupling Bernoulli invariants;
  - degrees of freedom allow to shift β with a constant.
- For typical equation of state for ISO3 and ISO4, continuity of β is equivalent to continuity of p.
- Alternative pH hierarchy for ISO2-4 with no gravity:
  - simpler and easier to implement;
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Conclusions

Outline

- We have written the Euler equations and other gas pipe models as pHDAE.
- Obtained energy-preserving interconnection of general gas pipe networks.

What next?

- Study, extend and implement pH-consistent numerical methods for discretization and model order reduction of gas networks.
- Develop a port-GENERIC formulation for the TA hierarchy, to grasp the thermodynamics better.

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