Relative energy estimates, asymptotic stability and structure preserving discretization for isentropic flow in gas networks

Herbert Egger, Jan Giesselmann, Teresa Kunkel and Nora Philippi

Numerical Analysis und Scientific Computing
Department of Mathematics, TU Darmstadt

Trends in Mathematical Modelling, Simulation and Optimisation: Theory and Applications

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Outline

1. Instationary gas flow in a single pipe

   dissipative Hamiltonian structure, parabolic limit problem

2. Stability analysis via relative energy estimates

   perturbations in data and parameters, asymptotic convergence, extension to networks

3. Structure preserving discretization

   mixed FEM, convergence estimates via relative energy, asymptotic stability

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Gas flow through a long pipe

Gas transport model

$$a\partial_t \rho + \partial_x(a\rho v) = 0$$

$$\partial_t(a\rho v) + \partial_x(a\rho v^2 + ap(\rho)) = -\tfrac{\lambda}{2D} a\rho v|v|$$

with $x \in (0, \ell)$ and $t > 0$

- a cross-sectional area, D diameter of pipe, λ friction coefficient
- ρ gas density, v velocity
- pressure law $p = p(\rho)$

see e.g. [BrouwerGasserHerty'13]

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Model equations

$$a\partial_\tau \rho + \partial_x m(\rho, w) = 0$$

$$\varepsilon^2 \partial_\tau w + \partial_x h(\rho, w) = -\gamma|w|w$$

- state variables: $\rho,\ w$
- co-state variables:

  mass flow rate $m(\rho, w) = a\rho w$

  total specific enthalpy $h(\rho, w) = \frac{\varepsilon^2}{2} w^2 + P'(\rho)$

with pressure potential $P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}\, dr$ such that $\frac{1}{\rho}\partial_x p(\rho) = \partial_x P'(\rho)$

Note: Parabolic limit problem in high friction regime for $\varepsilon \to 0$

Transformation to dissipative Hamiltonian system

Reformulated model equations

(1) $a\partial_\tau \rho + \partial_x m(\rho, w) = 0$

(2) $\varepsilon^2 \partial_\tau w + \partial_x h(\rho, w) = -\gamma |w| w$

Total energy of the system:

$$\mathcal{H}(\rho, w) = \int_0^\ell a\left(\tfrac{1}{2}\varepsilon^2 \rho w^2 + P(\rho)\right) dx$$

$$\Rightarrow \quad \frac{\delta \mathcal{H}}{\delta \rho} = ah, \; \frac{\delta \mathcal{H}}{\delta w} = \varepsilon^2 m$$

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Variational formulation: Multiply (1)–(2) with suitable test functions $q, \ r$, integrate over pipe $(0, \ell)$ with notation $(u, v) := \int_0^\ell uv\, dx$ to get

$$(a\partial_\tau \rho, q) + (\partial_x m, q) = 0 \qquad \forall q \in Q$$

$$(\varepsilon^2 \partial_\tau w, r) - (h, \partial_x r) + (\gamma \tfrac{|w|}{a\rho} m, r) = -hr|_0^\ell \qquad \forall r \in R$$

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Abstract dissipative Hamiltonian system

$$\mathcal{C}\partial_\tau \boldsymbol{u} + (\mathcal{J} + \mathcal{R}(\boldsymbol{u}))\boldsymbol{z}(\boldsymbol{u}) = \mathcal{B}_\partial \boldsymbol{z}(\boldsymbol{u})$$

$$\boldsymbol{z}(\boldsymbol{u}) = \mathcal{C}^{-1}\mathcal{H}'(u)$$

- state variables $\boldsymbol{u} = (\rho, w)$
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Power balance:

$$\frac{d}{dt}\mathcal{H}(\rho, w) = \langle \mathcal{C}\partial_\tau \boldsymbol{u}, \mathcal{C}^{-1}\mathcal{H}'(\boldsymbol{u})\rangle = -\langle \mathcal{R}(\boldsymbol{u})\boldsymbol{z}(\boldsymbol{u}), \boldsymbol{z}(\boldsymbol{u})\rangle + \langle \mathcal{B}_\partial \boldsymbol{z}(\boldsymbol{u}), \boldsymbol{z}(\boldsymbol{u})\rangle$$

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Abstract system:

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Perturbed system:

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Stability analysis on abstract level

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- Measure distance between $\boldsymbol{u}$ and $\widehat{\boldsymbol{u}}$ by relative energy see e.g. [Dafermos'16]

$$\mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) = \mathcal{H}(\boldsymbol{u}) - \mathcal{H}(\widehat{\boldsymbol{u}}) - \langle \mathcal{H}'(\widehat{\boldsymbol{u}}), \boldsymbol{u} - \widehat{\boldsymbol{u}} \rangle$$

- Energy identity:

$$\frac{d}{dt}\mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) = -\langle \mathcal{R}(\boldsymbol{u})\boldsymbol{z}(\boldsymbol{u}) - \mathcal{R}(\widehat{\boldsymbol{u}})\boldsymbol{z}(\widehat{\boldsymbol{u}}), \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}) \rangle + \langle \widehat{\boldsymbol{r}}, \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}) \rangle$$
$$+ \langle \mathcal{B}_\partial(\boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}})), \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}) \rangle + \langle C\partial_\tau \widehat{\boldsymbol{u}}, \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}) - \mathcal{C}^{-1}\mathcal{H}''(\widehat{\boldsymbol{u}})(\boldsymbol{u} - \widehat{\boldsymbol{u}}) \rangle$$

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- **Assume** that right hand side can be estimated by

$$rhs \leq C\,\mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) - \mathcal{D}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) + \mathcal{P}(\widehat{\boldsymbol{r}}) + \mathcal{P}_\partial(\boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}))$$

  with relative dissipation functional $\mathcal{D}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) \geq 0$ and perturbation functionals $\mathcal{P},\ \mathcal{P}_\partial$

- Use Gronwall to obtain stability estimate

Stability analysis on abstract level

Main assumptions:

(C1) $$-\langle \mathcal{R}(\boldsymbol{u})\boldsymbol{z}(\boldsymbol{u}) - \mathcal{R}(\widehat{\boldsymbol{u}})\boldsymbol{z}(\widehat{\boldsymbol{u}}), \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}})\rangle \leq C_1\mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) - 2\mathcal{D}(\boldsymbol{u}|\widehat{\boldsymbol{u}})$$

(C2) $$\langle C\partial_\tau \widehat{\boldsymbol{u}}, \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}) - \mathcal{C}^{-1}\mathcal{H}''(\widehat{\boldsymbol{u}})(\boldsymbol{u} - \widehat{\boldsymbol{u}})\rangle \leq C_2\mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}})$$

(C3) $$\langle \widehat{\boldsymbol{r}}, \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}})\rangle \leq C_3\mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) + \mathcal{D}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) + \mathcal{P}(\widehat{\boldsymbol{r}})$$

(C4) $$\langle \mathcal{B}_\partial(\boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}})), \boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}})\rangle \leq \mathcal{P}_\partial(\boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}(\widehat{\boldsymbol{u}}))$$

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with relative dissipation functional $\mathcal{D}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) \geq 0$ and perturbation functionals $\mathcal{P},\ \mathcal{P}_\partial$.

Additionally, assume **norm bounds**

(C0) $$c_0\|\boldsymbol{u} - \widehat{\boldsymbol{u}}\|_{\mathcal{C}}^2 \leq \mathcal{H}(\boldsymbol{u}|\widehat{\boldsymbol{u}}) \leq C_0\|\boldsymbol{u} - \widehat{\boldsymbol{u}}\|_{\mathcal{C}}^2$$

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Main result:

$$c_0\|\boldsymbol{u}(\tau) - \widehat{\boldsymbol{u}}(\tau)\|_{\mathcal{C}}^2 + \int_0^\tau e^{c(\tau-\sigma)}\mathcal{D}(\boldsymbol{u}|\widehat{\boldsymbol{u}})\, d\sigma$$
$$\leq \mathcal{C}_0\|\boldsymbol{u}(0) - \widehat{\boldsymbol{u}}(0)\|_{\mathcal{C}}^2 + \int_0^\tau e^{c(\tau-\sigma)}\big(\mathcal{P}(\widehat{\boldsymbol{r}}(\sigma)) + \mathcal{P}_\partial(\boldsymbol{z}(\boldsymbol{u}(\sigma)) - \boldsymbol{z}(\widehat{\boldsymbol{u}}(\sigma)))\big)\, d\sigma$$

Perturbation in parameters $\hat{\varepsilon}, \hat{\gamma}$, initial and boundary data

Variational formulation:

$$(a\partial_\tau \rho, q) + (\partial_x m, q) = 0 \qquad \forall q \in Q$$
$$(\varepsilon^2 \partial_\tau w, r) - (h, \partial_x r) + (\gamma \tfrac{|w|}{a\rho} m, r) = -h_\partial r|_0^\ell \qquad \forall r \in R$$

Perturbed formulation:

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- Inserting solution of perturbed problem $(\hat{\rho}, \hat{w})$ in original equations yields residual

$$\widehat{\boldsymbol{r}}_1 = 0, \quad \widehat{\boldsymbol{r}}_2 = (\varepsilon^2 - \hat{\varepsilon}^2)(\partial_\tau \hat{w} + \tfrac{1}{2}\partial_x |\hat{w}|^2) + (\gamma - \hat{\gamma}) \tfrac{|\hat{w}|}{a\hat{\rho}} \hat{w}$$

- Verify assumptions (C0) – (C4) for gas flow problem to obtain stability result

- Perturbation in parameters $\hat{\varepsilon}, \ \hat{\gamma}$, initial and boundary data
- Show abstract conditions (C0)–(C4) under following reasonable assumptions

Main assumptions

(A1) Bounds on parameters $0 < \varepsilon, \hat{\varepsilon}, \leq \bar{\varepsilon}, \ 0 < \underline{\gamma} \leq \gamma, \hat{\gamma} \leq \bar{\gamma}$

(A2) Smooth solutions $(\rho, w), \ (\hat{\rho}, \hat{w})$ with bounded states, i.e.,

$$0 < \underline{\rho} \leq \rho, \hat{\rho} \leq \bar{\rho}, \ |w|, |\hat{w}| \leq \bar{w}$$

(A3) Subsonic flow, i.e., $P(\rho)$ smooth, strongly convex with $\rho P''(\rho) \geq 4\bar{\varepsilon}^2 \bar{w}^2$

H. Egger, J. Giesselmann (2020): Stability and asymptotic analysis for instationary gas transport via relative energy estimates. arXiv:2012.14135.

Application to gas transport

- Perturbation in parameters $\hat{\varepsilon}$, $\hat{\gamma}$, initial and boundary data
- Show abstract conditions (C0)–(C4) under following reasonable assumptions

Main assumptions

(A1) Bounds on parameters $0 < \varepsilon, \hat{\varepsilon}, \leq \bar{\varepsilon},\ 0 < \underline{\gamma} \leq \gamma, \hat{\gamma} \leq \bar{\gamma}$

(A2) Smooth solutions $(\rho, w),\ (\hat{\rho}, \hat{w})$ with bounded states, i.e.,

$$0 < \underline{\rho} \leq \rho, \hat{\rho} \leq \bar{\rho},\ |w|, |\hat{w}| \leq \bar{w}$$

(A3) Subsonic flow, i.e., $P(\rho)$ smooth, strongly convex with $\rho P''(\rho) \geq 4\bar{\varepsilon}^2 \bar{w}^2$

Important ingredients

- (C0) Uniform norm bounds $\|\rho - \hat{\rho}\|_{L^2}^2 + \varepsilon^2 \|w - \hat{w}\|_{L^2}^2 \leq c_0^{-1} \mathcal{H}(\rho, w|\hat{\rho}, \hat{w})$
- (C1) Estimation of dissipation term

$$-(\gamma(|w|w - |\hat{w}|\hat{w}), m - \hat{m}) \leq \frac{2\bar{\gamma}|\bar{w}|^3}{\underline{\rho} c_0} \mathcal{H}(\rho, w|\hat{\rho}, \hat{w}) - \frac{1}{4} a \underline{\gamma} \underline{\rho} \|w - \hat{w}\|_{L^3}^3$$

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Application to gas transport

Main result:

$$||\rho(\tau) - \hat{\rho}(\tau)||^2_{L^2} + \varepsilon^2||w(\tau) - \hat{w}(\tau)||^2_{L^2} + \int_0^\tau ||w(s) - \hat{w}(s)||^3_{L^3}\, ds$$
$$\leq Ce^{c\tau}\Big(||\rho(0) - \hat{\rho}(0)||^2_{L^2} + \varepsilon^2||w(0) - \hat{w}(0)||^2_{L^2}$$
$$+ |\gamma - \hat{\gamma}|^{3/2} + |\varepsilon^2 - \hat{\varepsilon}^2| + \int_0^\tau |h_\partial(s) - \hat{h}_\partial(s)|_\partial\Big)$$

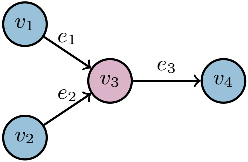
where c, C depending only on bounds in assumption and on $\partial_\tau\hat{\rho},\ \partial_\tau\hat{w}$.

Consequences:

- Stability of sub-sonic bounded state solutions with respect to perturbations in model parameters and initial and boundary data
- Uniqueness: $\hat{\varepsilon} = \varepsilon,\ \hat{\gamma} = \gamma,\ \hat{h}_\partial = h_\partial \ \Rightarrow\ \hat{\rho} = \rho,\ \hat{w} = w$
- Quantitative estimates for parabolic limit $\varepsilon \to 0$

H. Egger, J. Giesselmann (2020): Stability and asymptotic analysis for instationary gas transport via relative energy estimates. arXiv:2012.14135.

- Network given by directed graph with pipes $e \in \mathcal{E}$ and vertices $v \in \mathcal{V}$
- Gas transport equations hold on each pipe $e \in \mathcal{E}$



Coupling at pipe junctions:

- conservation of mass: $\sum_{e \in \mathcal{E}(v)} m^e(v) n^e(v) = 0$
- continuity of total specific enthalpy: $h^e(v) = h^v$

Network = sum of pipes + coupling conditions: $(\cdot,\cdot)_{\mathcal{E}} = \sum_{e \in \mathcal{E}} (\cdot,\cdot)_e$

Extension to networks

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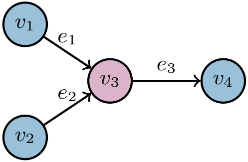

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Network = sum of pipes + coupling conditions: $(\cdot,\cdot)_{\mathcal{E}} = \sum_{e \in \mathcal{E}} (\cdot,\cdot)_e$

Variational formulation on network:

$$(a\partial_\tau \rho, q)_{\mathcal{E}} + (\partial_x m, q)_{\mathcal{E}} = 0 \qquad \forall q \in Q$$

$$(\varepsilon^2 \partial_\tau w, r)_{\mathcal{E}} - (h, \partial_x r)_{\mathcal{E}} + (\gamma \frac{|w|}{a\rho} m, r)_{\mathcal{E}} = -(h_\partial^v, r\, n)_{\mathcal{V}_\partial} \qquad \forall r \in R$$

- Variational formulation has same structure as on single pipe
- **Consequence:** Stability result directly transfers to networks!

Outline

1. Instationary gas flow in a single pipe

   dissipative Hamiltonian structure, parabolic limit problem

2. Stability analysis via relative energy estimates

   perturbations in data and parameters, asymptotic convergence, extension to networks

3. Structure preserving discretization

   mixed FEM, convergence estimates via relative energy, asymptotic stability

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1. Instationary gas flow in a single pipe

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   mixed FEM, convergence estimates via relative energy, asymptotic stability

Fully discrete scheme: Find $\rho_h \in Q_h \subset Q,\ m_h \in R_h \subset R$ so that

$$(\bar{\partial}_\tau \rho_h^n, q_h)_\mathcal{E} + (\partial_x m_h^n, q_h)_\mathcal{E} = 0 \qquad \forall q_h \in Q_h$$

$$(\varepsilon^2 \bar{\partial}_\tau \tilde{w}_h^n, r_h)_\mathcal{E} - (\tilde{h}_h^n, \partial_x r_h)_\mathcal{E} + (\gamma |\tilde{w}_h^n| w_h^n, r_h)_\mathcal{E} = -(h_\partial^v, r\, n)_{\mathcal{V}_\partial} \qquad \forall r_h \in R_h$$

with $\tilde{w}_h^n := \frac{m_h^n}{a \rho_h^n},\ \tilde{h}_h^n := \frac{1}{2}\varepsilon^2 \frac{|m_h^n|^2}{a^2 |\rho_h^n|^2} + P'(\rho_h^n),$ and $\bar{\partial}_\tau \rho_h^n = \frac{1}{\Delta\tau}(\rho_h^n - \rho_h^{n-1})$.

- Based on variational formulation, mixed FEM in space, implicit Euler in time
- Stability analysis via relative energy estimates transfers to discrete problem

In preparation: H. Egger, J. Giesselmann, T. Kunkel, N. Philippi: An asymptotic preserving Galerkin scheme for instationary gas transport in pipe networks.

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- Based on variational formulation, mixed FEM in space, implicit Euler in time
- Stability analysis via relative energy estimates transfers to discrete problem

Consequences:

- Discrete stability and asymptotic preserving scheme for $\varepsilon \to 0$
- Quantitative convergence rates

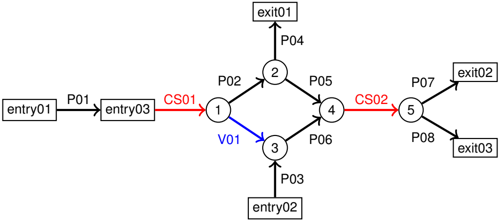
  $\rightarrow$ choose perturbed solution $\hat{\rho},\ \hat{w}$ as projections of exact solution $\rho,\ w$ in analysis

In preparation: H. Egger, J. Giesselmann, T. Kunkel, N. Philippi: An asymptotic preserving Galerkin scheme for instationary gas transport in pipe networks.

Transient scenario on GasLib-11

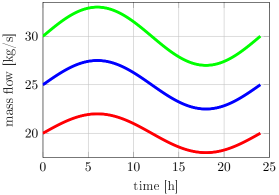
see [GasLib - A Library of Gas Network Instances]

- Network parameters from 'GasLib-11.net'
- $p(\rho) = c^2\rho$ with $c = 343\mathrm{m/s}$
- Compressor stations and valves handled as pipes of length 0
- Time interval of 24h
- Initial data given by steady state



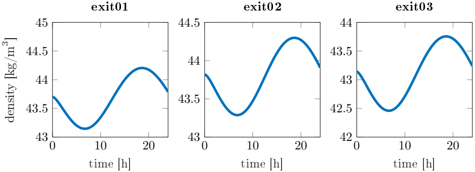
Boundary conditions:

- Constant enthalpy h at entries
- Mass flow m at exit01, exit02, exit03



Discretization error

- $\varepsilon = 1$, relative error w.r.t. L^2-norm at $t = 24\text{h}$
- Elements per edge/ time steps: $2^i,\ i = 2, .., 10$



Error in ρ	Rate	Error in m	Rate
1.57e-3	-	4.31e-3	-
7.74e-4	1.02	1.89e-3	1.19
3.84e-4	1.01	8.22e-4	1.20
1.91e-4	1.02	3.72e-4	1.14
9.54e-5	1.00	1.76e-4	1.08
4.77e-5	1.00	8.51e-5	1.05
2.38e-5	1.00	4.19e-5	1.02
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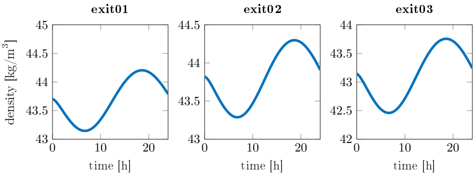

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2.38e-5	1.00	4.19e-5	1.02
1.19e-5	1.00	2.08e-5	1.01

Asymptotic convergence

- Relative error to solution for $\varepsilon = 0$ w.r.t. L^2-norm at $t = 24\mathrm{h}$
- 32 elements per edge, 500 time steps

	ε = 1	ε = 1e-1	ε = 1e-2	ε = 1e-3	ε = 1e-4	ε = 1e-5
Error in density ρ	5.64e-5	5.64e-7	5.64e-9	5.64e-11	5.64e-13	5.82e-15
Error in mass flow m	3.18e-5	3.19e-07	3.19e-09	3.19e-11	3.17e-13	2.25e-14

Observation: Can compute gas transport using parabolic problem!

Instationary gas flow and stability analysis[1]

- Suitable (rescaled) variational formulation has dissipative Hamiltonian structure
- Relative energy estimates yield stability w.r.t. perturbations, asymptotic convergence, uniqueness for sub-sonic, bounded state solutions of gas equations
- **Extension to networks:** Structure is conserved, stability results directly transfer

Structure preserving discretization[2]

- Based on variational formulation using mixed FEM in space, implicit Euler in time
- Stability analysis via relative energy estimates directly transfers to discrete problem: Discrete stability, asymptotic preserving, quantitative convergence rates
- Numerical tests indicate that modeling with parabolic problem is appropriate

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[1] H. Egger, J. Giesselmann (2020): Stability and asymptotic analysis for instationary gas transport via relative energy estimates. arXiv:2012.14135.

[2] In preparation: H. Egger, J. Giesselmann, T. Kunkel, N. Philippi: An asymptotic preserving Galerkin scheme for instationary gas transport in pipe networks.