Subproject A01

Global Methods for Stationary Gastransport

TRR 154

Mathematical modeling, simulation, and optimization using the example of gas networks

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Main Objective

Given a gas network as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, we want to find global solutions of optimization problems of the form

$$
\begin{aligned}
\min \quad & c(p, q, z) \\
\text{s.t.} \quad & \sum_{a \in \delta^+(u)} q_a - \sum_{a \in \delta^-(u)} q_a = q_u^{\pm} && \forall u \in \mathcal{V}, \\
& f_a(x, p_a(x), \partial_x p_a(x), q) = 0 && \forall a \in \mathcal{A}_{pipe} \subseteq \mathcal{A}, \\
& g_a(p_a, q, z) \le 0 && \forall a \in \mathcal{A} \setminus \mathcal{A}_{pipe}, \\
& p_u = p_a(0), \ \ p_v = p_a(L_a) && \forall a = (u,v) \in \mathcal{A}_{pipe}, \\
& \underline{q}_a \le q_a \le \overline{q}_a && \forall a \in \mathcal{A}, \\
& \underline{p}_u \le p_u \le \overline{p}_u && \forall u \in \mathcal{V}, \\
& z_a \in \{0, 1\} && \forall a \in \mathcal{Z} \subseteq \mathcal{A}.
\end{aligned}
$$

Here:

- ▷ f_a can be the stationary Euler equation (ISO1)

$$\partial_x p(x) \left(1 - \frac{c^2 q^2}{A^2 p(x)^2}\right) = -\frac{\lambda c^2}{2DA^2} q|q| \frac{1}{p(x)} - \frac{g}{c^2} s\, p(x).$$

- ▷ g_a represents the models of the other network elements:
  - ◦ linear models for (control)valves, resistors, compressors,
  - ◦ nonlinear models for resistors, compressors.
- ▷ z_a are discrete decisions, e.g., open/close valve.

Solution Approach

Idea: Replace $f_a = 0$ by the relation $F_{in}(p_{out}, q) = p_{in}$.

- ▷ *Problem:* We cannot compute F_{in} exactly.
- ▷ *Solution:*
  - ◦ By choosing two numerical methods with, respectively, nonpositive and nonnegative local truncation error, we can compute lower and upper bounds for F_{in}. In our case, we choose the modified Euler's method and the trapezoidal rule.
  - ◦ Assuming $s = 0$, $q > 0$ and $4Ap > 5cq$, i.e., the gas travels with subsonic velocity, we define

$$h(p, q) := -\frac{\lambda c^2 q^2 p}{2D\,(A^2 p^2 - c^2 q^2)}.$$

Thus, $\partial_x p(x) = h\big(p(x), q\big)$ holds.

  - ◦ After dividing $[0, L]$ into N equidistant steps, we compute

$$p_N^e = p(L), \quad p_{i-1}^e = p_i^e - \Delta x\, h\Big(p_i^e - \tfrac{\Delta x}{2} h(p_i^e, q), q\Big),$$
$$p_N^t = p(L), \quad p_{i-1}^t = p_i^t - \tfrac{1}{2}\Delta x \left[h(p_{i-1}^t, q) + h(p_i^t, q)\right]$$

for $i = N, \dots, 1$ and $\Delta x := L/N$.

  - ◦ Then we define $p_{in}^e(p_N^e, q) := p_0^e$ and $p_{in}^t(p_N^t, q) := p_0^t$, such that the following inequality holds:

$$p_{in}^e(p_N^e, q) \ \le\ p(0) \ \le\ p_{in}^t(p_N^t, q).$$

  - ◦ Hence, we can replace $f_a = 0$ with

$$p_{in}^e(p_v, q_a) \ \le\ p_u \ \le\ p_{in}^t(p_v, q_a).$$

The Functions p_{in}^e, p_{in}^t

- ▷ Assuming $\Delta x < 0.16 \frac{D}{\lambda}$, the functions p_{in}^e and p_{in}^t are:
  - ◦ monotone increasing w.r.t. p and q,
  - ◦ convex in (p, q),
  - ◦ and continuously differentiable.

  These are the same properties as F_{in} has.
- ▷ Via adapting Newton's method:
  - ◦ Solve trapezoidal rule with a tolerance of 10^{-5}Pa.
  - ◦ Solution is an upper bound on the exact solution, i.e., $p(0) \le p_0^t$ is guaranteed.
- ▷ Gradient cuts for p_{in}^e are valid inequalities if orientation of flow is fixed. (The relation $F_{in}(p_{out}, q) = p_{in}$ is convex.)

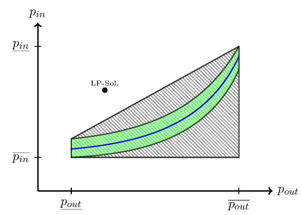
Algorithmic Procedure

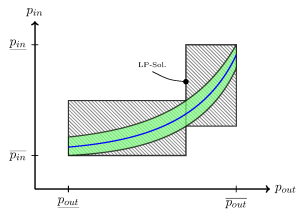
We use a (spatial) Branch-and-Cut approach with the framework SCIP. In every node we perform the following steps:

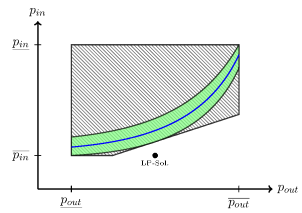
1. Bound Propagation
2. Solve relaxation (i.e., model without the ODEs)
3. For every $a = (u, v) \in \mathcal{A}_{pipe}$ check if solution (p_u, p_v, q_a) is feasible, i.e.,

$$
\begin{aligned}
p_{in}^e(p_v, q_a) \ &\le\ p_u \ \le\ p_{in}^t(p_v, q_a) \quad \text{if } q_a \ge 0, \\
p_{in}^e(p_u, q_a) \ &\le\ p_v \ \le\ p_{in}^t(p_u, q_a) \quad \text{if } q_a < 0.
\end{aligned}
$$

   If $p_{in}^t(p_v, q_a) - p_{in}^e(p_v, q_a) \ \ge\ \delta$ or $p_{in}^t(p_u, q_a) - p_{in}^e(p_u, q_a) \ \ge\ \delta$ holds, we increase N and recompute the schemes.
4. Choose the most infeasible pipe and either
   - a) fix orientation of flow,
   - b) add a concave overestimator,
   - c) perform branching,
   - d) or add a gradient cut.







Upcoming Work

- ▷ Convergence analysis of the algorithm.
- ▷ Extend theory to non-horizontal pipes, i.e., $s \neq 0$.
- ▷ Identify minimal requirements, such that this approach works.

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