Subproject B01

TRR 154

Mathematical modeling, simulation, and optimization using the example of gas networks

Adaptive Dynamic Multiscale Methods

Pascal Mindt   Pia Domschke   Jens Lang

TECHNISCHE UNIVERSITÄT DARMSTADT

Technische Universität Darmstadt

www.tu-darmstadt.de

Summary

The aim of this project is the development of a dynamic multiscale approach for the numerical solution of the compressible instationary Euler equations on networks. The modelling is based on the one-dimensional Euler equations. With physically simplified models these equations form a model hierachy in a natural manner. Using adjoint-based a posteriori error estimators, on accuracy-controlled adaptive simulation of practicaly relevant gas networks is realized. Core points are an efficient control of spatial and temporal discretizations as well as the model hierachy to create a basis for multilevel optimization.

Euler Equations

▷ The temperature-dependent instationary Euler equations are

$$\begin{cases} U_t + F(U) &= G(U), \ (x,t) \in [a,b] \times (0,T], \ 0 < T, \ a < b \\ U(x,0) &= U_0(x), \ x \in [a,b] \end{cases}$$

with

$$U \equiv \begin{pmatrix} \rho \\ q \\ E \end{pmatrix}, \quad F(U) \equiv \begin{pmatrix} q \\ \frac{q^2}{\rho} + p \\ \frac{q}{\rho}(E+p) \end{pmatrix}, \quad G(U) \equiv \begin{pmatrix} 0 \\ -\frac{\lambda q |q|}{2D\rho} \\ -\frac{k_w}{D}(T - T_w) \end{pmatrix} \tag{1}$$

▷ The variables ρ, q, E, p, T stand for *density, mass flux, total energy, pressure* and *temperature* respectively. It holds $E = \frac{p}{\gamma - 1} + \frac{q^2}{\rho}$ and $T = \frac{p}{R\rho}$ to close the system.

▷ Further definitions: γ *adiabatic coefficient*, λ *friction coefficient*, *D pipe diameter*, k_w *heat exchange rate*, T_w *outer temperature*, *R gas constant*.

Finite Volume Approach

Integration of (1) over the intervals $I_i \equiv [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$: $[a,b] = \bigcup_{i=1}^{M} I_i$ leads to the semi-discrete ODE-system

$$\partial_t \bar{U}_i(t) = \mathcal{L}_i(\mathbf{U})(t) \equiv -\frac{1}{\Delta x}\left(\hat{F}_{i+\frac{1}{2}}(t) - \hat{F}_{i-\frac{1}{2}}(t)\right) + G_i(t), \ i = 1, ..., M, \tag{2}$$

with

▷ Finite volume $\mathbf{U}(t) \equiv \left(\bar{U}_i(t)\right)_{i=1}^{M}$, $\bar{U}_i(t) \equiv \frac{1}{\Delta x}\int_{I_i} U(x,t)\, dx$,

▷ Numerical flux function $\hat{F}_{i+\frac{1}{2}}(\mathbf{U})(t) \approx F(U(x_{i+\frac{1}{2}}, t)$

▷ Numerical integration of the source $G_i(t) \approx \frac{1}{\Delta x}\int_{I_i} G(U(x,t))\, dx$

▷ Spatial grid size $\Delta x \equiv x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$,

Numerical Treatment of the Euler Equations

▷ Involving the *weighted essentially non-oscilatory* procedure $\mathcal{W}$

$$\partial_t \mathbf{U}(t) = \mathcal{L}\left(\mathcal{W}(\mathbf{U})\right)(t) \tag{3}$$

increases the spatial order, cp. [1].

▷ (3) is then solved numerically on the time grid

$$[t^n, t^{n+1}]_{n=1}^{N} : \bigcup_{n=1}^{N} [t^n, t^{n+1}], \text{ with } \Delta t \equiv t^{n+1} - t^n$$

by an *s*-stage *singly diagonally implicit Runge-Kutta* method of order 2 (SDIRK(2, *s*))

$$\begin{cases} Y_j &= \mathbf{U}^n + \Delta t \sum_{k=1}^{s} a_{j,k}\mathcal{L}(\mathcal{W}(Y_k)), \ 1 \le j \le s \\ \mathbf{U}^{n+1} &= \mathbf{U}^n + \Delta t \sum_{k=1}^{s} b_k \mathcal{L}(\mathcal{W}(Y_k)) \end{cases} \tag{4}$$

with $\mathbf{U}^n \approx \mathbf{U}(t^n)$ and the optimal parameters

$$a_{k,k} = \frac{1}{2s}, \ a_{j,k} = \frac{1}{2s}, \ j < k, \ b_k = \frac{1}{s}.$$

▷ Whenever $(\mathcal{L}, \|\cdot\|)$ has a contractive *forward Euler* step (FE)

$$\|\mathbf{U} + \Delta_{FE}\mathcal{L}(\mathbf{U})\| \le \|\mathbf{U}\|$$

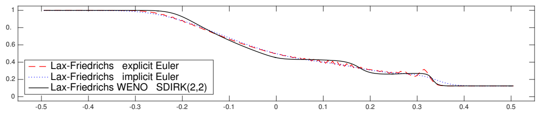
(4) provides a *strong stability preserving* solution in $\|\cdot\|$, cp. [2]

$$\|Y_j\| \le \|\mathbf{U}^n\|, \ \forall \Delta t \le 2s\Delta t_{FE}, \ 1 \le j \le s.$$

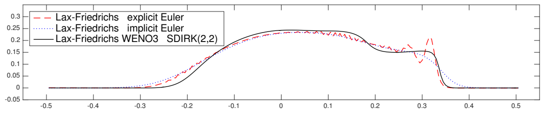
Contribution to Demonstrator 3

Simulation of temperature dependent Euler equations

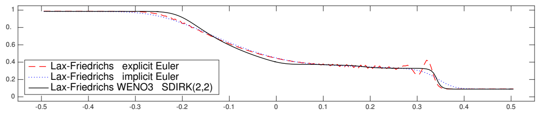
▷ Density



▷ Mass flux



▷ Energy



References

[1] G. Jiang, C. Shu. Efficient implementation of weighted ENO schemes. Technical report, DTIC Document, 1995

[2] M.N. Spijker. Stepsize conditions for general monotonicity in numerical initial value problems. *SIAM Journal on Numerical Analysis*, 45(3): 1226-1245, 2007

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

TECHNISCHE UNIVERSITÄT DARMSTADT

UNIVERSITÄT DUISBURG ESSEN Open-Minded