Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

An a posteriori Probabilistic Robustness Check for Deterministic Optimal Controls

Michael Schuster[1], **Elisa Strauch**[2], **Jens Lang**[2], **Martin Gugat**[1]

2023 September 25, EUCCO 2023, Heidelberg

[1] Friedrich-Alexander Universität Erlangen-Nürnberg (FAU),
[2] Technische Universität Darmstadt (TUD)

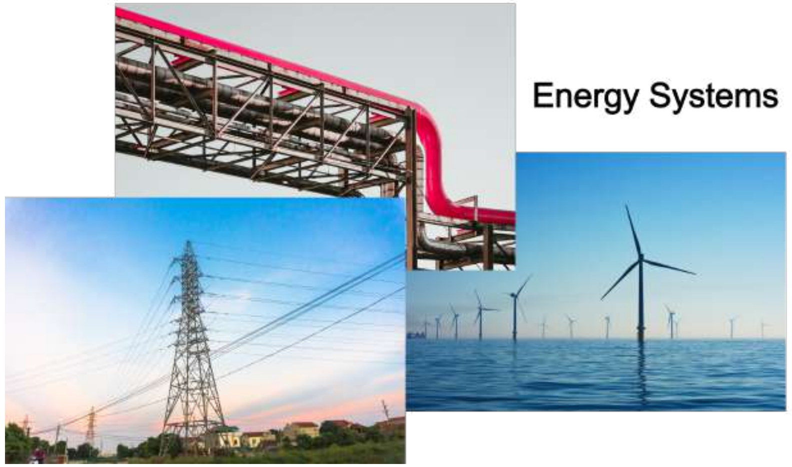
Motivation

Uncertainty in Dynamical Systems

Medicine & Epidemiology

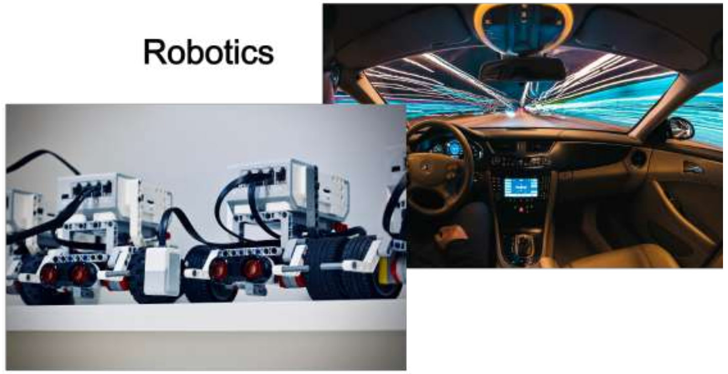
Aerospace Engineering

Uncertainty often plays a significant role in dynamical systems!



Energy Systems

Robotics



Pictures from *https://unsplash.com/de*

Motivation

Robust vs Probabilistic

Robust Optimization:

Consider the optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g(x,u) \leq 0 \quad \forall u \in \mathcal{U} \end{aligned}$$

with objective function f, constraint g, decision vector x, parameter u and uncertainty set $\mathcal{U}$.

Probabilistic Constrained Optimization:

Consider the optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & \mathbb{P}(\, g(x,\xi) \leq 0 \,) \; \geq \; \alpha \end{aligned}$$

with objective function f, constraint g, decision vector x, random variable ξ and probability level α.

Gas Network Optimization

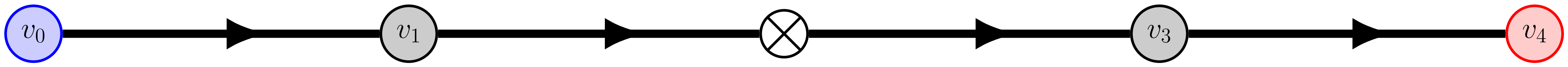
Mathematical Modelling

p	gas pressure	v	gas velocity	g	gravitational constant
ρ	gas density	λ/D	pipe friction	α	pipe slope

Isothermal Euler Equations

$$\rho_t + (\rho v)_x = 0$$

$$(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha)$$



Source: Inlet Pressure

$$p(t) = p_0(t)$$

Compressor Station

$$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$$

Sink: Gas Outflow

$$(\rho v)(t) = b(t)$$

Coupling Conditions

Conservation of Mass: $\sum(\rho v)_{\text{in}}(t) = \sum(\rho v)_{\text{out}}(t),$ Continuity in Pressure: $p_{\text{in}} = p_{\text{out}}$

The Optimal Control System

Let bounds for the pressures $0 < p_{\min} < p_{\max}$ be given at every node. Consider the optimal control problem

$$\min_{u \in L^2(0,T)} \quad f(u)$$

s.t.

$$\rho_t + (\rho v)_x = 0 \qquad \text{on every edge}$$

$$(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v|v| - g\rho\sin(\alpha)$$

$$p(t) = p_0(t) \qquad \text{on every source node}$$

$$(\rho v)(t) = b(t) \qquad \text{on every sink node}$$

$$\sum (\rho v)_{\text{in}}(t) = \sum (\rho v)_{\text{out}}(t) \qquad \text{on every inner node}$$

$$p_{\text{in}} = p_{\text{out}}$$

$$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t) \qquad \text{for every compressor}$$

$$p(t) \in \left[\, p_{\min},\ p_{\max} \,\right] \quad \forall t \in [0,T] \qquad \text{on every node}$$

Gas Network Optimization

Uncertainty and Probabilistic Robustness

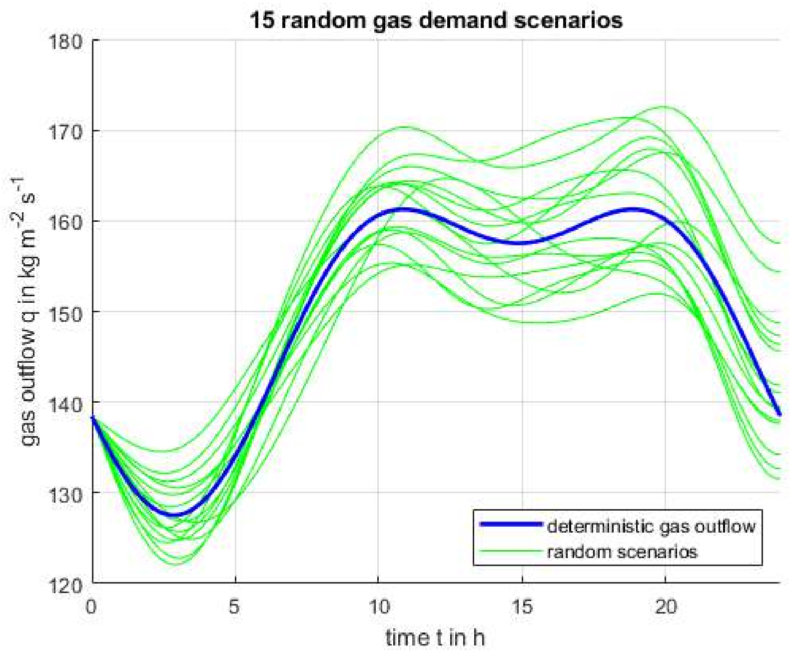
We a posteriori assume that the consumers gas demand $b(t)$ is random:

- Write boundary condition $b(t)$ as Fourier series

$$b(t) = \sum_{m=0}^{\infty} a_m^0(f)\ \psi_m(t)$$

- For random variables $\xi_m \sim \mathcal{N}(1, \sigma)$ define

$$b^\omega(t) = \sum_{m=0}^{\infty} \xi_m(\omega)\ a_m^0(b)\ \psi_m(t)$$



Probabilistic Robustness

For a control u^*, the probability

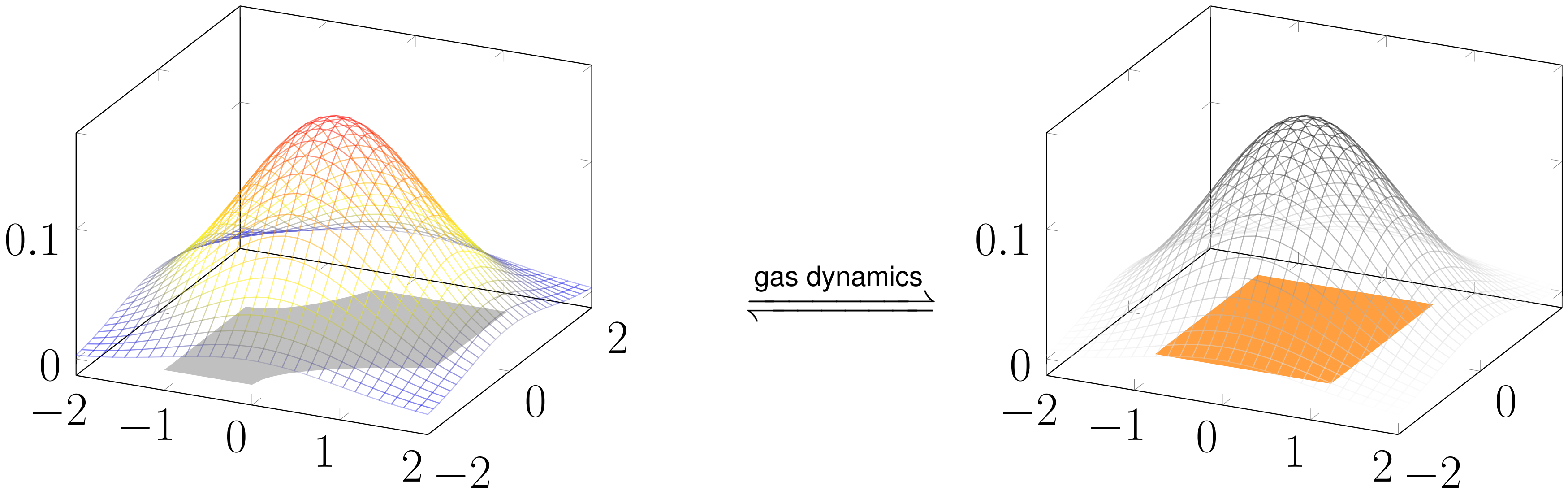
$$\mathbb{P}\big(\ b^\omega \in M(u^*, t) \quad \forall\, t \in [0, T]\ \big)$$

is called probabilistic robustness of u^*, where $M(u^*, t)$ contains all constraints of the optimal control problem.

Gas Network Optimization

Uncertainty and Probabilistic Robustness

$$\mathbb{P}\big(\, b^\omega \in M(u^*, t) \quad \forall\, t \in [0,T] \,\big) \;=\; \mathbb{P}\big(\, p^\omega(t) \in [\, p_{\min},\; p_{\max} \,] \quad \forall\, t \in [0,T] \,\big)$$



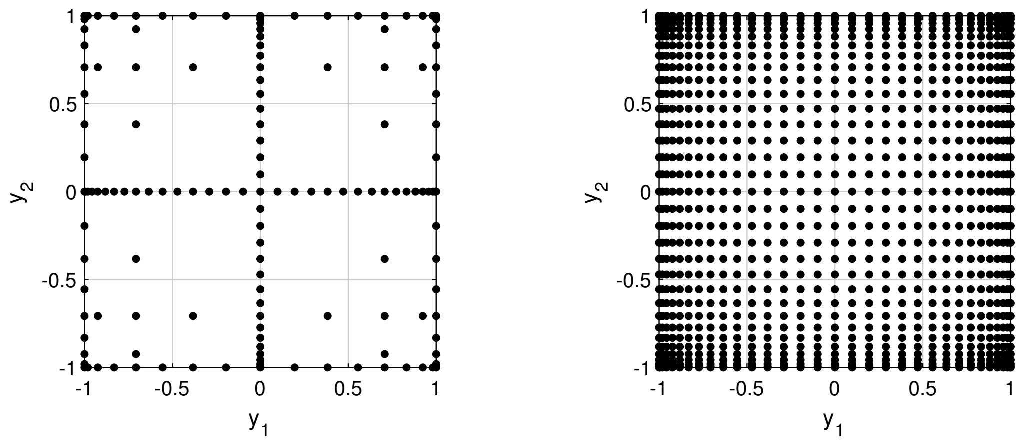
(a) Well-known distribution (colored), unknown set of feasible loads (grey)

(b) Unknown distribution (grey), well-known set of feasible pressures (orange)

Probabilistic Robustness

Stochastic Collocation

- Let an optimal control $u(t)$, inlet pressures $p_0(x)$ and a n_d-dimensional random gas outflow be given
- We approximate the pressure $p(t, x, b^\omega)$ in the stochastic space by stochastic collocation on a *Smolyak sparse grid* with *Clenshaw-Curtis* nodes



- For a multi-index $\mathbf{i} \in \mathbb{N}^{n_S}$ and a natural number $k \in \mathbb{N}$ we define the constant

$$c_{\mathbf{i}} = (-1)^{k+n_S-|\mathbf{i}|} \binom{n_S - 1}{k + n_S - |\mathbf{i}|}.$$

- The approximated pressures are given by the Smolyak formula with level $k > 0$

$$S_k\big[p(t, x, \,\cdot\,)\big] = \sum_{\substack{\mathbf{i} \,\in\, \mathbb{N}^{n_S} \\ k+1 \,\leq\, \mathbf{i} \,\leq\, k+n_S}} c_{\mathbf{i}} \left(\mathcal{U}^{(i_1)} \otimes \cdots \otimes \mathcal{U}^{(i_{n_S})}\right) \big[p(t, x, \,\cdot\,)\big],$$

where the $\mathcal{U}^{(i_m)}$ for $m = 1, \cdots, n_S$ are the interpolation operators on one dimension.

Probabilistic Robustness

Kernel Density Estimation

- Let $\mathcal{B} = \{\ b^{S,1}(t), \cdots, b^{S,N_{\text{KDE}}}(t)\ \}$ be independent and identically distributed random boundary functions
- Let $\mathcal{P}_{\mathcal{B}} = \{\ p(t; b^{S,1}), \cdots, p(t; b^{S,N_{\text{KDE}}})\ \}$ be the corresponding densities at the end of the pipe (given by the approximation of the solution in the stochastic space)

$$\mathbb{P}\left(\, p^{\omega}(t) \in [\, p_{\text{min}},\ p_{\text{max}}\,] \quad \forall t \in [0, T]\,\right) \quad = \quad \mathbb{P}\begin{pmatrix} \min\limits_{t\in[0,T]} p^{\omega}(t) \in [\, p_{\text{min}},\ p_{\text{max}}\,] \\ \max\limits_{t\in[0,T]} p^{\omega}(t) \in [\, p_{\text{min}},\ p_{\text{max}}\,] \end{pmatrix}$$

- Let $\left\{ \begin{bmatrix} \underline{p}(b_1) := \min\limits_{t\in[0,T]} p(t; b^{S,1}) \\ \overline{p}(b_1) := \max\limits_{t\in[0,T]} p(t; b^{S,1}) \end{bmatrix}, \cdots, \begin{bmatrix} \underline{p}(b_{N_{\text{KDE}}}) := \min\limits_{t\in[0,T]} p(t; b^{S,N_{\text{KDE}}}) \\ \overline{p}(b_{N_{\text{KDE}}}) := \max\limits_{t\in[0,T]} p(t; b^{S,N_{\text{KDE}}}) \end{bmatrix} \right\} \subseteq \mathbb{R}^{2d}$ be a sample of the minimal and maximal densities in $[0, T]$

Kernel density estimator with Gaussian kernels for bandwidths h^{min} and h^{min}

$$\varrho_{p,N_{\text{KDE}}}(z) = \frac{1}{N_{\text{KDE}}\ \prod_{j=1}^{d} h_j^{\text{min}}\ h_j^{\text{max}}}\ \frac{1}{(2\pi)^d} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{d} \exp\left(-\frac{1}{2}\left(\frac{z_{1,j} - \underline{p}_j(b_i)}{h_j^{\text{min}}}\right)^2\right) \cdot \exp\left(-\frac{1}{2}\left(\frac{z_{2,j} - \overline{p}_j(b_i)}{h_j^{\text{max}}}\right)^2\right)$$

Probabilistic Robustness

Kernel Density Estimation

$$\mathbb{P}\left(p^{\omega}(t) \in [\, p_{\mathsf{min}},\ p_{\mathsf{max}} \,] \quad \forall t \in [0,T] \right) = \int_{[p_{\min},p_{\max}]^d} \varrho_{p,N_{\mathsf{KDE}}}(z)\, dz$$

$$= \int_{[p_{\min},p_{\max}]^d} \frac{1}{N_{\mathsf{KDE}} \prod_{j=1}^{d} h_j^{\mathsf{min}}\, h_j^{\mathsf{max}}} \frac{1}{(2\pi)^d} \sum_{i=1}^{N_{\mathsf{KDE}}} \prod_{j=1}^{d} \exp\left(- \left(\frac{z_{1,j} - \underline{p}_j(b_i)}{\sqrt{2}\, h_j^{\mathsf{min}}} \right)^2 \right) \cdot \exp\left(- \left(\frac{z_{2,j} - \overline{p}_j(b_i)}{\sqrt{2}\, h_j^{\mathsf{max}}} \right)^2 \right) dz$$

Probabilistic Robustness

Kernel Density Estimation

$$\mathbb{P}\left(\, p^{\omega}(t) \in [\, p_{\text{min}},\ p_{\text{max}}\,] \quad \forall t \in [0,T]\,\right) = \int\limits_{[p_{\min},p_{\max}]^d} \varrho_{p,N_{\text{KDE}}}(z)\, dz$$

$$= \int\limits_{[p_{\min},p_{\max}]^d} \frac{1}{N_{\text{KDE}}\ \prod_{j=1}^{d} h_j^{\text{min}}\ h_j^{\text{max}}} \frac{1}{(2\pi)^d} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{d} \exp\left(-\left(\frac{z_{1,j} - \underline{p}_j(b_i)}{\sqrt{2}\ h_j^{\text{min}}}\right)^2\right) \cdot \exp\left(-\left(\frac{z_{2,j} - \overline{p}_j(b_i)}{\sqrt{2}\ h_j^{\text{max}}}\right)^2\right) dz$$

$$= \frac{1}{N_{\text{KDE}}\ \prod_{j=1}^{d} h_j^{\text{min}}\ h_j^{\text{max}}} \frac{1}{(2\pi)^d} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{d} \int_{p_{\min}}^{p_{\max}} \exp\left(-\left(\frac{z_{1,j} - \underline{p}_j(b_i)}{\sqrt{2}\ h_j^{\text{min}}}\right)^2\right) dz_{1,j} \cdot \int_{p_{\min}}^{p_{\max}} \exp\left(-\left(\frac{z_{2,j} - \overline{p}_j(b_i)}{\sqrt{2}\ h_j^{\text{max}}}\right)^2\right) dz_{2,j}$$

Probabilistic Robustness

Kernel Density Estimation

$$\mathbb{P}\left(p^{\omega}(t) \in [\, p_{\text{min}},\ p_{\text{max}}\,] \quad \forall t \in [0, T] \right) = \int\limits_{[p_{\min}, p_{\max}]^d} \varrho_{p, N_{\text{KDE}}}(z)\, dz$$

$$= \int\limits_{[p_{\min}, p_{\max}]^d} \frac{1}{N_{\text{KDE}} \prod_{j=1}^{d} h_j^{\text{min}}\, h_j^{\text{max}}} \frac{1}{(2\pi)^d} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{d} \exp\left(- \left(\frac{z_{1,j} - \underline{p}_j(b_i)}{\sqrt{2}\, h_j^{\text{min}}} \right)^2 \right) \cdot \exp\left(- \left(\frac{z_{2,j} - \overline{p}_j(b_i)}{\sqrt{2}\, h_j^{\text{max}}} \right)^2 \right) dz$$

$$= \frac{1}{N_{\text{KDE}} \prod_{j=1}^{d} h_j^{\text{min}}\, h_j^{\text{max}}} \frac{1}{(2\pi)^d} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{d} \int\limits_{p_{\min}}^{p_{\max}} \exp\left(- \left(\frac{z_{1,j} - \underline{p}_j(b_i)}{\sqrt{2}\, h_j^{\text{min}}} \right)^2 \right) dz_{1,j} \cdot \int\limits_{p_{\min}}^{p_{\max}} \exp\left(- \left(\frac{z_{2,j} - \overline{p}_j(b_i)}{\sqrt{2}\, h_j^{\text{max}}} \right)^2 \right) dz_{2,j}$$

$$= \frac{1}{N_{\text{KDE}}\, 4^d} \sum_{i=1}^{N_{\text{KDE}}} \prod_{j=1}^{d} \left[\operatorname{erf}\left(\frac{p_{\max} - \underline{p}_j(b_i)}{\sqrt{2}\, h_j^{\min}} \right) - \operatorname{erf}\left(\frac{p_{\min} - \underline{p}_j(b_i)}{\sqrt{2}\, h_j^{\min}} \right) \right] \cdot \left[\operatorname{erf}\left(\frac{p_{\max} - \overline{p}_j(b_i)}{\sqrt{2}\, h_j^{\min}} \right) - \operatorname{erf}\left(\frac{p_{\min} - \overline{p}_j(b_i)}{\sqrt{2}\, h_j^{\min}} \right) \right]$$

Probabilistic Robustness

Optimization with Buffer Zones

Let bounds for the pressures $0 < p_{\min} < p_{\max}$ be given at every node. For $\varepsilon > 0$ consider the optimal control problem

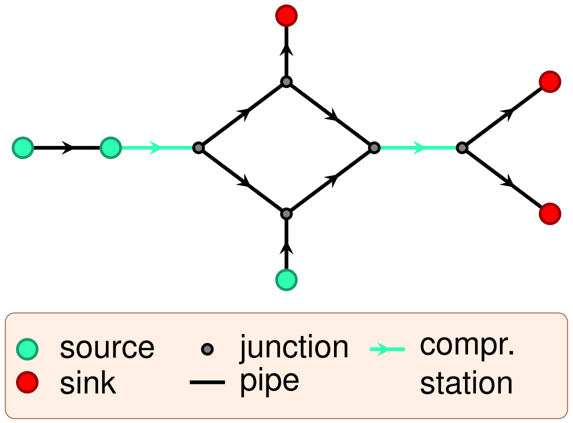
$$\min_{u \in L^2(0,T)} \quad f(u)$$

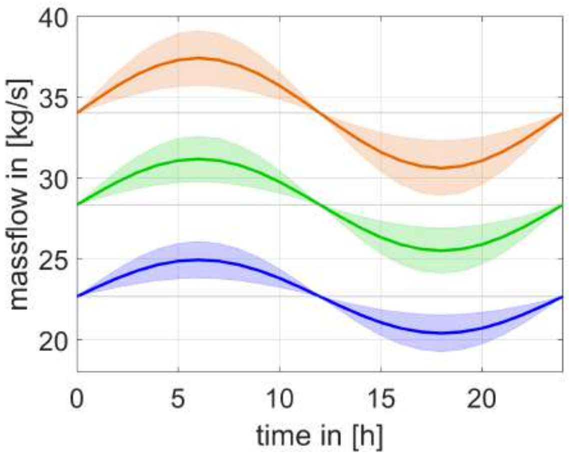
s.t.

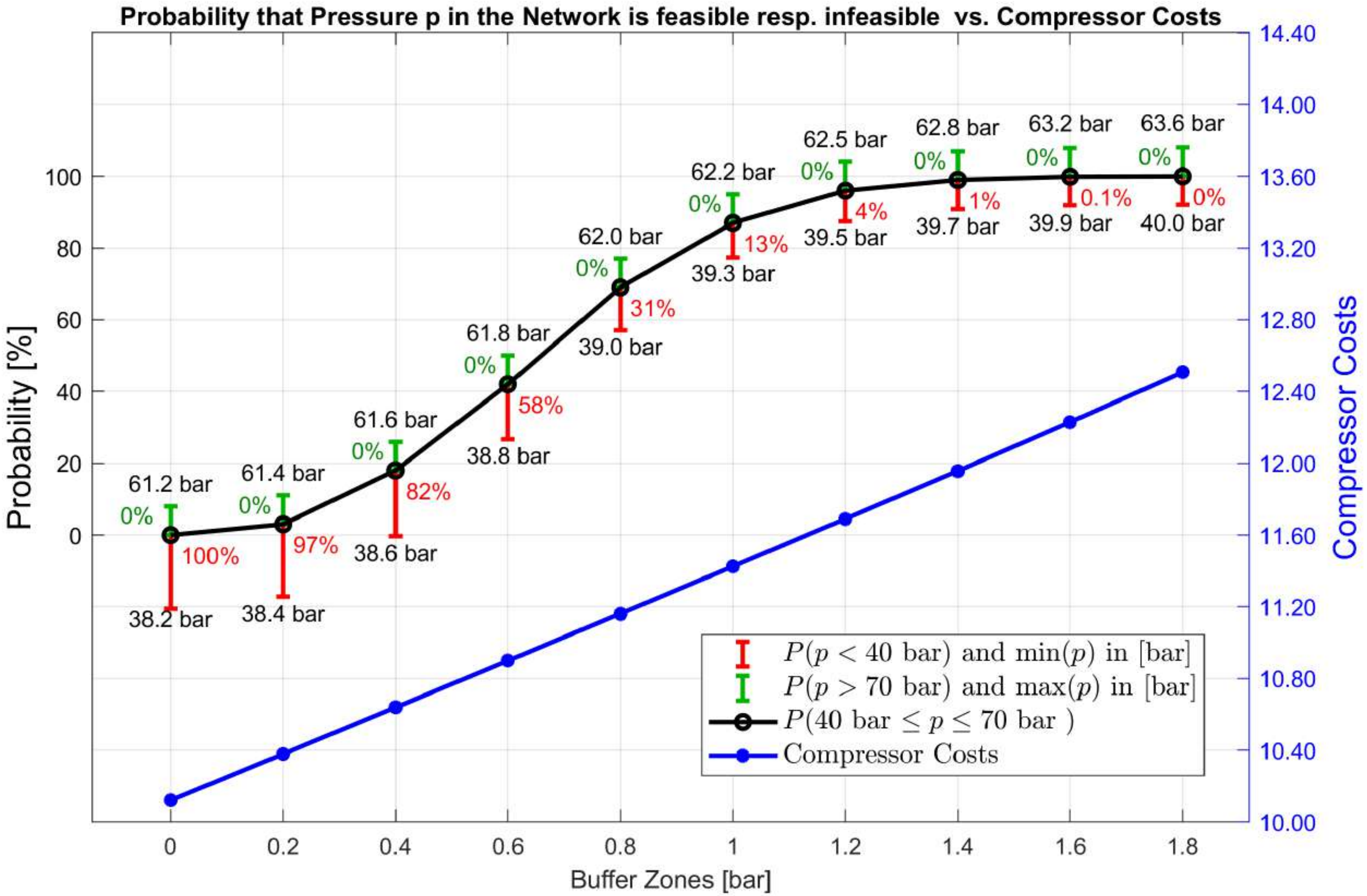
$\rho_t + (\rho v)_x = 0$ $(\rho v)_t + \left(p + \rho v^2\right)_x = -\frac{\lambda}{2D}\rho v\lvert v\rvert - g\rho\sin(\alpha)$	on every edge
$p(t) = p_0(t)$	on every source node
$(\rho v)(t) = b(t)$	on every sink node
$\sum (\rho v)_{\text{in}}(t) = \sum (\rho v)_{\text{out}}(t)$ $p_{\text{in}} = p_{\text{out}}$	on every inner node
$\frac{p_{\text{out}}(t)}{p_{\text{in}}(t)} = u(t)$	for every compressor
$p(t) \in \left[\, p_{\min} + \varepsilon,\ p_{\max} - \varepsilon \,\right] \quad \forall t \in [0, T]$	on every node

Numerical Example

Result for GasLib-11







A posteriori Probabilistic Robustness Check for Deterministic Controls

[Schuster, Strauch, Lang, Gugat, 2023]: *An a posteriori Probabilistic Robustness Check for Deterministic Optimal Controls.* Preprint

[Schuster, 2021]: *Nodal Control and Probabilistic Constrained Optimization using the Example of Gas Networks.* Dissertation at FAU Erlangen-Nürnberg

[Strauch, 2023]: *Adaptive Multi-Level Monte Carlo and Stochastic Collocation Methods for Hyperbolic Partial Differential Equations with Random Data on Networks.* Dissertation at TU Darmstadt