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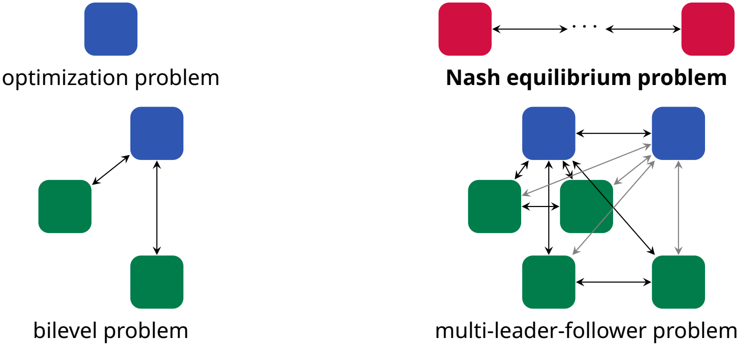
A Primer on Nash Equilibrium Problems

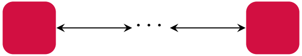
Terminology, Existence, Algorithms

Motivation and context



- Consider a system of **coupled** optimization problems.
- All optimization problems have to be solved **simultaneously**.
- Applications e.g. in **energy markets**, traffic flow, shared resource allocation.

Basic Terminology



Example: Cournot duopoly

Problem:

$$\max_{x_i \geq 0} \quad f_i(x_1, x_2) = p(x_1, x_2) \cdot x_i - c_i(x_i) \qquad \forall i = 1, 2$$

- We consider two firms $i = 1, 2$.
- Each firm i chooses its output $x_i \geq 0$ and tries to maximize its gain.
- The market price depends on the total supply, i.e. on $x_1 + x_2$.

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Equilibrium: A solution $x^* = (x_1^*, x_2^*)$ should satisfy

$$\begin{aligned} x_1^* &= \operatorname*{argmax}_{x_1 \geq 0} \ f_1(x_1, x_2^*), \\ x_2^* &= \operatorname*{argmax}_{x_2 \geq 0} \ f_2(x_1^*, x_2). \end{aligned}$$

Nash equilibrium problems (NEP)

Definition

A **Nash equilibrium problem (NEP)** consists of

- a set $\{1, \dots, N\}$ of finitely many **players**,
- **strategy sets** $X_i \subseteq \mathbb{R}^{n_i}$ for every player $i = 1, \dots, N$,
- **payoff functions** $f_i : X \to \mathbb{R}$ for every player $i = 1, \dots, N$.

Each player i tries to solve the problem

$$\min_{x_i} \quad f_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_N) \quad \text{s.t.} \quad x_i \in X_i.$$

Remarks:

- For theoretical purposes we consider only minimization problems and in economic applications often maximization problems.
- $X := X_1 \times \dots \times X_N \subseteq \mathbb{R}^n$ denotes the Cartesian product of all strategy sets.

Nash equilibria (NE)

Definition

A vector $x^* = (x_1^*, \dots, x_N^*)$ is called a **Nash equilibrium (NE)**, if for all players $i = 1, \dots, N$

- the strategy $x_i^* \in X_i$ is feasible and
- optimal in the sense that
$$f_i(x_i^*, x_{-i}^*) \leq f_i(x_i, x_{-i}^*) \quad \forall x_i \in X_i.$$

Remarks:

- In a Nash equilibrium, unilateral deviations do not improve a player's payoff.
- We use x_{-i} as a shorthand for the opponents' strategies, i.e. $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$.

Characterizing Nash equilibria using best responses

- Given opponents' strategies x_{-i}, we define the **best response map of player *i*** as
$$S_i(x_{-i}) = \operatorname*{argmin}_{x_i \in X_i} f_i(x_i, x_{-i}).$$
- Given a strategy vector x, the **best response map** is defined as
$$S(x) = (S_1(x_{-1}), \ldots, S_N(x_{-N})).$$

Lemma

A strategy vector $x^* = (x_1^*, \ldots, x_N^*)$ is a Nash equilibrium if and only if it is a fixed point of the best response map, meaning
$$x^* \in S(x^*).$$

Exercise: Nash equilibria of a Cournot duopoly

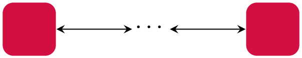
Consider the Cournot duopoly

$$\max_{x_i \geq 0} \quad f_i(x_1, x_2) = p(x_1, x_2) \cdot x_i - c \cdot x_i$$

with $p(x_1, x_2) = a - b(x_1 + x_2)$ and $a > c > 0$, $b > 0$ and compute all Nash equilibria (x_1^*, x_2^*).

Existence Results for NEPs

- Existence using Best Responses
- Existence using Variational Inequalities



Kakutani's fixed point theorem for set-valued maps

Theorem (Kakutani's fixed point theorem)

Let $X \subseteq \mathbb{R}^n$ be nonempty, compact and convex and $F : X \rightrightarrows X$ a closed set-valued map such that $F(x)$ is nonempty and convex for all $x \in X$. Then F has a fixed point, i.e. there is an $x^*\in X$ such that $x^* \in F(x^*)$.

Definition

Let $X \subseteq \mathbb{R}^n$. A set-valued map $F : X \rightrightarrows \mathbb{R}^n$ is called **closed**, if for all convergent sequences $(x^k)_k \to_X x^*$ and $(y^k)_k \to y^*$ with $y^k \in F(x^k)$ for all $k \in \mathbb{N}$ we have $y^* \in F(x^*)$.

Remarks:

- Kakutani's fixed point theorem is a generalization of Brouwer's fixed point theorem to set-valued maps.
- Closedness of the set-valued map replaces the continuity assumption in Brouwer's fixed point theorem.

Existence theorem of Nikaido and Isoda

Theorem (Existence theorem of Nikaido and Isoda)

Assume that the strategy sets $X_i \subseteq \mathbb{R}^{n_i}$ are nonempty, compact and convex and payoff functions $f_i : X \longrightarrow \mathbb{R}$ are continuous in x and (quasi)convex in x_i for every fixed x_{-i} for all $i = 1, \dots, N$. Then there exists at least one Nash equilibrium.

Remarks:

- A function $f : X \longrightarrow \mathbb{R}$ is **quasiconvex** on a convex set $X \subseteq \mathbb{R}^n$, if for all $x, y \in X$ and all $c \in (0, 1)$ we have
$$f(cx + (1-c)y) \le \max\{f(x), f(y)\}.$$
- Quasiconvexity is a generalization of convexity, which can be characterized by the property that all sublevel sets of f are convex.

Exercise: Existence theorem of Nikaido and Isoda

Prove the existence theorem of Nikaido and Isoda.

Interlude: Optimality condition for a single optimization problem

Problem:

$$\min_{x \in X} \quad f(x)$$

with $X \subseteq \mathbb{R}^n$ convex and $f : X \to \mathbb{R}$ continuously differentiable (on an open superset of X).

Lemma

(a) *Let $x^* \in X$ be a local minimum of f. Then*

$$\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in X.$$

(b) *Let f be (pseudo)convex on X and $x^* \in X$ with*

$$\nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in X.$$

Then x^* is a global minimum of f on X.

Variational inequality problem (VIP)

Definition

Let $X \subseteq \mathbb{R}^n$ be nonempty, closed, and convex and $F : X \to \mathbb{R}^n$. Then a **variational inequality problem (VIP)** is the task to find a solution $x^* \in X$ with

$$F(x^*)^T(x - x^*) \geq 0 \qquad \forall x \in X.$$

We denote this problem by $\text{VIP}(X, F)$.

Remarks:

- A continuously differentiable function $f : X \to \mathbb{R}$ is **pseudoconvex** on a convex set $X \subseteq \mathbb{R}^n$, if for all $x, y \in X$ we have
$$\nabla f(x)^T(y - x) \geq 0 \qquad \Longrightarrow \qquad f(y) \geq f(x).$$
- Solving a pseudoconvex optimization problem is equivalent to solving VIP($X, \nabla f$), see previous slide.

Exercise: KKT conditions and variational inequality problems

The **KKT conditions** for the nonlinear optimization problem

$$\min_x \quad f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0$$

with $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ are

$$\begin{aligned} \nabla f(x) + \nabla g(x)\lambda + \nabla h(x)\mu &= 0, \\ 0 \leq -g(x) \perp \lambda \geq 0, \qquad h(x) &= 0 \end{aligned}$$

with $x \in \mathbb{R}^n$ and multipliers $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$.

Rewrite the KKT conditions as a variational inequality problem.

Exercise: Variational inequality problems and projections

Let $X \subseteq \mathbb{R}^n$ be nonempty, closed, and convex and $F : X \longrightarrow \mathbb{R}^n$.

Show that x^* solves $\mathrm{VIP}(X, F)$ if and only if x^* is a fixed point of $H(x) := P_X(x - \gamma F(x))$ with $\gamma > 0$.

From Nash equilibrium problem to variational inequality problem

- Recall that in a NEP every player $i = 1, \ldots, N$ solves the problem
$$\min_{x_i \in X_i} f_i(x_i, x_{-i}).$$
- If X_i is convex and $x_i \mapsto f_i(x_i, x_{-i})$ is pseudoconvex for fixed x_{-i}, then the problem of player i is equivalent to the VIP
$$\nabla_{x_i} f_i(x_i, x_{-i})^T (x_i - x_i^*) \geq 0 \qquad \forall x_i \in X_i.$$

From Nash equilibrium problem to variational inequality problem

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- If X_i is convex and $x_i \mapsto f_i(x_i, x_{-i})$ is pseudoconvex for fixed x_{-i}, then the problem of player i is equivalent to the VIP
$$\nabla_{x_i} f_i(x_i, x_{-i})^T (x_i - x_i^*) \geq 0 \qquad \forall x_i \in X_i.$$
- Defining $X := X_1 \times \ldots \times X_N$ and
$$F(x) := \begin{pmatrix} \nabla_{x_1} f_1(x_1, x_{-1}) \\ \vdots \\ \nabla_{x_N} f_N(x_N, x_{-N}) \end{pmatrix},$$
we see that — under the given assumptions — the NEP is equivalent to VIP(X, F).

Remarks:

- Instead of having to solve N optimization problems simultaneously, we can solve one joint VIP.
- This observation is useful for existence results and solution algorithms.

Existence results for variational inequality problems

Theorem

Let $X \subseteq \mathbb{R}^n$ be nonempty, closed and convex and $F : X \longrightarrow \mathbb{R}^n$.

(a) *If F is continuous and X bounded, then $VIP(X, F)$ has at least one solution.*

(b) *If F is (pseudo)monotone, then the solution set of $VIP(X, F)$ is convex (possibly empty).*

(c) *If F is strictly monotone, then $VIP(X, F)$ has at most one solution (possibly none).*

(d) *If F is uniformly monotone, then $VIP(X, F)$ has at exactly one solution.*

Remarks:

- Part (a) follows from Brouwer's fixed point theorem.
- A function $F : X \longrightarrow \mathbb{R}^n$ is **monotone**, if for all $x, y \in X$ we have $(F(x) - F(y))^T (x - y) \geq 0$.
- If $F : X \longrightarrow \mathbb{R}^n$ is differentiable, then it is monotone if and only if $F'(x)$ is positive semidefinite on X.
- Uniform monotonicity is also called strong monotonicity (not to be confused with strict monotonicity).

Applying the existence results to Nash equilibrium problems

Recap: We are interested in the case $X := X_1 \times \ldots \times X_N$ and

$$F(x) := \begin{pmatrix} \nabla_{x_1} f_1(x_1, x_{-1}) \\ \vdots \\ \nabla_{x_N} f_N(x_N, x_{-N}) \end{pmatrix},$$

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Conditions on *X*:

- X is nonempty/convex/closed/bounded if and only if all X_i are nonempty/convex/closed/bounded.

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Conditions on *X*:

- X is nonempty/convex/closed/bounded if and only if all X_i are nonempty/convex/closed/bounded.

Conditions on *F*:

- If $x_i \mapsto f_i(x_i, x_{-i})$ is sufficiently smooth, then the following are equivalent:
  - $x_i \mapsto f_i(x_i, x_{-i})$ is convex on X_i
  - $x_i \mapsto \nabla_{x_i} f_i(x_i, x_{-i})$ is monotone on X_i
  - $x_i \mapsto \nabla^2_{x_i x_i} f_i(x_i, x_{-i})$ is positive semidefinite on X_i
- Unfortunately, monotonicity of the components $\nabla_{x_i} f_i(x_i, x_{-i})$ is not enough to ensure monotonicity of F.

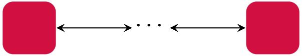
Exercise: Existence results for a Cournot oligopoly

Consider the Cournot oligopoly

$$\max_{x_i \geq 0} \quad f_i(x_1, x_2) = p(x) \cdot x_i - c_i \cdot x_i$$

with $p(x) = a - b \sum_{j=1}^{N} x_j$ and $a, b, c_i > 0$ for $i = 1, \ldots, N$. What do the existence results tell us about this NEP?

Generalization of NEPs



Example: Modified Cournot duopoly

Problem:

$$\max_{x_i} \quad f_i(x_1, x_2) = p(x_1, x_2) \cdot x_i - cx_i \quad \text{s.t.} \quad x_i \geq 0, \quad x_1 + x_2 \leq C$$

with $p(x_1, x_2) = a - b(x_1 + x_2)$ and $a, b, c, C > 0$.

Examples for such shared constraints are

- shared network capacities,
- shared emission caps,
- market clearing constraints.

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Challenge: Shared or coupled constraints to not fit in our current NEP framework.

Generalized Nash equilibrium problems

Definition

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- a set $\{1, \ldots, N\}$ of finitely many **players**,
- **strategy set maps** $X_i : \mathbb{R}^{n-n_i} \rightrightarrows \mathbb{R}^{n_i}$ for every player $i = 1, \ldots, N$,
- **payoff functions** $f_i : \mathbb{R}^n \to \mathbb{R}$ for every player $i = 1, \ldots, N$.

Each player i tries to solve the problem

$$\min_{x_i} \quad f_i(x_i, x_{-i}) \quad \text{s.t.} \quad x_i \in X_i(x_{-i}).$$

Remarks:

- For a given strategy vector x define the set of **feasible reactions** as $\Omega(x) := X_1(x_{-1}) \times \ldots \times X_N(x_{-N})$.
- Only fixed points $x^* \in \Omega(x^*)$ can be solutions of the GNEP.
- Shared constraints result in a joint feasible set $X \subseteq \mathbb{R}^n$ such that $X_i(x_{-i}) = \{x_i \mid (x_i, x_{-i}) \in X\}$.

Exercise: Feasible reactions on a joint feasible set

Consider a GNEP with $N = 2$ players and the joint feasible set

$$X = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}.$$

Determine the set of feasible reactions $\Omega(x)$ to the following strategy vectors:

$$x = (0, 0), \qquad x = (0, 1), \qquad x = (-1, -1)$$

Normalized Nash equilibria on a joint feasible set

Definition

Consider a generalized Nash equilibrium problem with N players and a joint feasible set X. A point $x^* \in X$ is called a **normalized Nash equilibrium**, if it satisfies

$$\Psi(x^*, y) = \sum_{i=1}^{N} f_i(x_i^*, x_{-i}^*) - f_i(y_i, x_{-i}^*) \geq 0 \qquad \forall y \in X.$$

Remarks:

- The function Ψ is called the **Nikaido-Isoda function** and can also be used in solution algorithms.
- A Nash equilibrium $x^* \in X$ is characterized by $\Psi(x^*, y) \geq 0$ for all $y \in \Omega(x^*)$.
- For NEPs, there is no difference between Nash equilibria and normalized Nash equilibria.
- For GNEPs with a joint feasible set X, normalized Nash equilibria form a subset of its Nash equilibria.

Exercise: Normalized Nash equilibria and multipliers

Consider a GNEP with N players, payoff functions f_i and a joint feasible set

$$X = \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}.$$

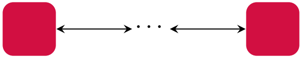
Assume that a suitable CQ holds in $x^* \in X$. Compare the resulting KKT systems for

(a) x^* being a Nash equilibrium of the GNEP,

(b) x^* being a normalized Nash equilibrium of the GNEP.

Some Solution Approaches

- Algorithms for NEPs
- Algorithms for VIPs
- Algorithms for GNEPs



Recap: Nash equilibrium problems (NEPs)

Definition

A **Nash equilibrium problem (NEP)** consists of N coupled optimization problems of the form

$$\min_{x_i} \quad f_i(x_i, x_{-i}) \quad \text{s.t.} \quad x_i \in X_i.$$

A **Nash equilibrium (NE)** is a strategy vector $x^* \in X = X_1 \times \ldots \times X_N$ with

$$x_i^* \in \operatorname*{argmin}_{x_i \in X_i} \ f_i(x_i, x_{-i}^*) \qquad \forall i = 1, \ldots, N.$$

Jacobi- and Gauss-Seidel method

Idea:

- Iteratively determine best responses of all players
$$x_i^{k+1} = \operatorname*{argmin}_{x_i \in X_i} \; f_i(x_i, x_{-i}^k).$$
- Jacobi relies on parallel updates, Gauss-Seidel on sequential updates.

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- Jacobi relies on parallel updates, Gauss-Seidel on sequential updates.

Benefits:

- Easy to implement.
- Distributed/asynchronous version of Jacobi method possible.

Downsides:

- Convergence typically not guaranteed, iterates can cycle or diverge.
- Feasibility not guaranteed for GNEPs.

Exercise: Gauss-Seidel method

Apply the Gauss-Seidel method to the following two NEPs:

(a) rock, paper, scissors

(b) Cournot duopoly with $a > c > 0$, $b > 0$:

$$\min_{x_i \geq 0} f_i(x_i, x_{-i}) = a - b \cdot (x_1 + x_2) - c \cdot x_i \qquad \forall i = 1, 2$$

Recap: Variational inequality reformulation of NEPs

- Consider a NEP
$$\min_{x_i} \; f_i(x_i, x_{-i}) \quad \text{s.t.} \quad x_i \in X_i \qquad \forall i = 1, \ldots, N$$
with $X_i \subseteq \mathbb{R}^{n_i}$ convex and $x_i \mapsto f_i(x_i, x_{-i})$ pseudoconvex for fixed x_{-i}.
- Then NEP is equivalent to $\text{VIP}(X, F)$ with $X := X_1 \times \ldots \times X_N$ and
$$F(x) := \begin{pmatrix} \nabla_{x_1} f_1(x_1, x_{-1}) \\ \vdots \\ \nabla_{x_N} f_N(x_N, x_{-N}) \end{pmatrix}.$$

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with $X_i \subseteq \mathbb{R}^{n_i}$ convex and $x_i \mapsto f_i(x_i, x_{-i})$ pseudoconvex for fixed x_{-i}.
- Then NEP is equivalent to VIP(X, F) with $X := X_1 \times \ldots \times X_N$ and
$$F(x) := \begin{pmatrix} \nabla_{x_1} f_1(x_1, x_{-1}) \\ \vdots \\ \nabla_{x_N} f_N(x_N, x_{-N}) \end{pmatrix}.$$
- Solving a variational inequality VIP(X, F) with $X \subseteq \mathbb{R}^n$ nonempty, closed, convex and $F : X \longrightarrow \mathbb{R}^n$ is equivalent to computing a fixed point of
$$H(x) := P_X(x - \gamma F(x))$$
with $\gamma > 0$ arbitrary.

Projection-based algorithms for VIPs

Idea:

- With some $\gamma > 0$ iteratively compute the projections
$$x^{k+1} := H(x^k) = P_X(x^k - \gamma F(x^k)).$$
- Variations use halfsteps to update $F(x^k)$ before updating x^k or vary the parameter γ_k.

Projection-based algorithms for VIPs

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- Variations use halfsteps to update $F(x^k)$ before updating x^k or vary the parameter γ_k.

Benefits:

- Requires only evaluations of F itself, no derivatives.
- In case $X := X_1 \times \ldots \times X_N$ the projections can be computed separately.

Downsides:

- Convergence can require strong assumptions concerning monotonicity and Lipschitz continuity of F.
- Convergence is typically slow.
- Computing the projections can be computationally expensive.

Exercise: Projection method

Consider a NEP with $X_i \subseteq \mathbb{R}^n$ be nonempty, closed and convex and $x_i \mapsto f_i(x_i, x_{-i})$ be (pseudo)convex. How can the iterates generated by the projection method applied to the corresponding VIP(X, F) be interpreted in terms of the original NEP?

Some other algorithms for VIPs

Gap functions:

- The function $g : X \longrightarrow \mathbb{R} \cup \{\infty\}$ with
$$g(x) := \sup_{y \in X} F(x)^T (x - y)$$
has the property $g(x) \geq 0$ on X and $g(x^*) = 0$ if and only if x^* solves $\text{VIP}(X, F)$.
- So one can solve $\text{VIP}(X, F)$ by minimizing g on X.
- In order to ensure differentiability, one typically has to add regularization term to g.

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- In order to ensure differentiability, one typically has to add regularization term to g.

Generalized KKT conditions:

- Assume $X = \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\}$ with $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^p$ continuously differentiable.
- If x^* solves $\mathrm{VIP}(X, F)$ and a CQ for X holds in x^*, then x^* solves the KKT system
$$\begin{aligned} &F(x) + \nabla g(x)\lambda + \nabla h(x)\mu = 0, \\ &0 \leq \lambda \perp g(x) \leq 0, \qquad h(x) = 0. \end{aligned}$$

Exercise: Generalized KKT conditions

Prove that — in case a CQ for X holds in x^* — a solution x^* of VIP(X, F) solves the generalized KKT system.

Under which assumptions does a solution x^* of the generalized KKT system solve VIP(X, F)?

Recap: Generalized Nash equilibrium problems (GNEPs)

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A **generalized Nash equilibrium problem (GNEP)** consists of N coupled optimization problems of the form

$$\min_{x_i} f_i(x_i, x_{-i}) \quad \text{s.t.} \quad x_i \in X_i(x_{-i}).$$

A **generalized Nash equilibrium (GNE)** is a strategy vector $x^* \in \Omega(x^*)$ with

$$x_i^* \in \operatorname*{argmin}_{x_i \in X_i(x^*_{-i})} f_i(x_i, x^*_{-i}) \qquad \forall i = 1, \ldots, N.$$

Augmented Lagrangian method for GNEPs

Idea:

- Assume that strategy sets are given by

$$X_i(x_{-i}) = \{x_i \in \mathbb{R}^{n_i} \mid G_i(x_i, x_{-i}) \le 0, \quad H_i(x_i, x_{-i}) = 0\}$$

  and rewrite players' problems using the augmented Lagrangian with $\lambda_i \in \mathbb{R}^{m_i}, \mu_i \in \mathbb{R}^{p_i}$ and $\alpha_i > 0$:

$$\min_{x_i \in \mathbb{R}^{n_i}} f_i(x_i, x_{-i}) + \frac{\alpha_i}{2} \left[\left\| \max\{0, G_i(x_i, x_{-i}) + \tfrac{\lambda_i}{\alpha_i}\} \right\|_2^2 + \left\| H_i(x_i, x_{-i}) + \tfrac{\mu_i}{\alpha_i} \right\|_2^2 \right]$$

- Iteratively solve the resulting NEP, update the multipliers λ_i, μ_i and increase the penalty α_i, if needed.

Augmented Lagrangian method for GNEPs

Idea:

- Assume that strategy sets are given by
$$X_i(x_{-i}) = \{x_i \in \mathbb{R}^{n_i} \mid G_i(x_i, x_{-i}) \leq 0, \quad H_i(x_i, x_{-i}) = 0\}$$
and rewrite players' problems using the augmented Lagrangian with $\lambda_i \in \mathbb{R}^{m_i}, \mu_i \in \mathbb{R}^{p_i}$ and $\alpha_i > 0$:
$$\min_{x_i \in \mathbb{R}^{n_i}} f_i(x_i, x_{-i}) + \frac{\alpha_i}{2} \left[\left\| \max\{0, G_i(x_i, x_{-i}) + \tfrac{\lambda_i}{\alpha_i}\} \right\|_2^2 + \left\| H_i(x_i, x_{-i}) + \tfrac{\mu_i}{\alpha_i} \right\|_2^2 \right]$$
- Iteratively solve the resulting NEP, update the multipliers λ_i, μ_i and increase the penalty α_i, if needed.

Benefits:

- Can be applied to general GNEPs, not just ones with a joint feasible set.
- Individual constraints $g_i(x_i) \leq 0$, $h_i(x_i) = 0$ can be left as constraints.

Downsides:

- Each iteration requires the solution of a NEP.

Some other algorithms for GNEPs

Nikaido-Isoda function for GNEPs with a joint feasible set X:

- The function $V : X \longrightarrow \mathbb{R} \cup \{\infty\}$ with
$$V(x) := \sup_{y \in \Omega(x)} \Psi(x, y) = \sup_{y \in \Omega(x)} \sum_{i=1}^{N} f_i(x_i, x_{-i}) - f_i(y_i, x_{-i})$$
has the property $V(x) \geq 0$ on X and $V(x^*) = 0$ if and only if x^* is a GNE.
- So one can solve the GNEP by minimizing V on X.
- For better properties of V the set $\Omega(x)$ is often replaced with X and a regularization term is added.

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- So one can solve the GNEP by minimizing V on X.
- For better properties of V the set $\Omega(x)$ is often replaced with X and a regularization term is added.

Quasi-Variational inequalities (QVIP):

- In case $X_i(x_{-i})$ are convex and $x_i \mapsto f_i(x_i, x_{-i})$ are (pseudo)convex for fixed x_{-i}, solving a GNEP is equivalent to finding $x^* \in \Omega(x^*)$ with

$$F(x^*)^T (x - x^*) \geq 0 \qquad \forall x \in \Omega(x^*).$$

- So one can use algorithms for QVIPs to solve a GNEP.

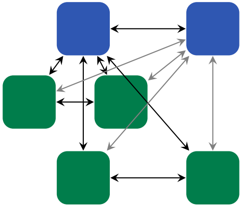
Exercise: Regularized Nikaido-Isoda function

Consider a GNEP with a nonempty, closed and convex joint feasible set $X \subseteq \mathbb{R}^n$ and convex functions $x_i \mapsto f_i(x_i, x_{-i})$ for all $i = 1, \dots, N$. For $\gamma > 0$ we define the regularized value function as

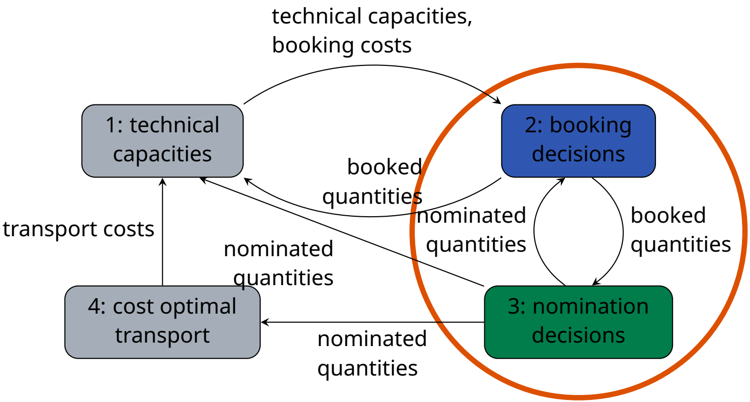
$$V_\gamma(x) := \sup_{y \in X} \quad \Psi(x, y) - \frac{\gamma}{2}\|x - y\|_2^2 = \sup_{y \in X} \quad \sum_{i=1}^{N} f_i(x_i, x_{-i}) - f_i(y_i, x_{-i}) - \frac{\gamma}{2}\|x - y\|_2^2.$$

Show that $V_\gamma(x) \geq 0$ for all $x \in X$ and $V_\gamma(x^*) = 0$ if and only if $x^* \in X$ is a normalized Nash equilibrium.

Outlook: Application in Gas Markets



Focus on the strategic decisions of gas sellers



Multi-leader multi-follower model

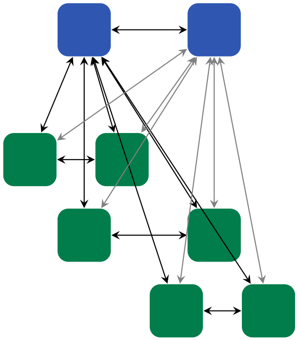
Consider competing firms $i = 1, \ldots, N$ and time periods $t = 1, \ldots, T$. Before the first time period, each firm can invest in capacity expansion and then every time period has to decide how much to produce/sell:

- **Capacity decision** before the first time period:

$$\begin{aligned} \max_{\text{capacity}_i} \quad & \left(\sum_{t=1}^{T} \text{weight}_t \cdot \text{production gain}_{t,i} \right) - \text{capacity cost}_i \cdot \text{capacity}_i \\ \text{s.t.} \quad & \text{capacity}_i \geq 0 \end{aligned}$$

- **Production decision** in each time period t:

$$\begin{aligned} \max_{\text{production}_{t,i}} \quad & \text{equilibrium price}_t \cdot \text{production}_{t,i} - \text{production cost}_{t,i} \\ \text{s.t.} \quad & 0 \leq \text{production}_{t,i} \leq \text{capacity}_i \end{aligned}$$



Lower-level problems: Capacity-constrained Cournot problems

Assume that the equilibrium price and production costs are linear:

$$P_t(Y) = \theta_t - bY \qquad \text{and} \qquad c_i(y_{t,i}) = c_i \cdot y_{t,i}.$$

Then the **lower-level problem** of firm i in time period t is given by

$$\max_{y_{t,i}} \quad P_t(y_{t,i}, y_{t,-i}) \cdot y_{t,i} - c_i \cdot y_{t,i} \quad \text{s.t.} \quad 0 \leq y_{t,i} \leq x_i.$$

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It is known that for all capacities $x = (x_1, \ldots, x_N) \geq 0$ in each time period $t = 1, \ldots, T$ the lower-level problem has a **unique Nash equilibrium** $\hat{y}_t(x)$ with equilibrium strategies

$$\hat{y}_{t,i}(x) = \begin{cases} 0 & \text{if firm } i \text{ is } \textbf{inactive}, \\ \frac{\hat{P}_t(x) - c_i}{b} \in (0, x_i) & \text{if firm } i \text{ is } \textbf{unconstrained}, \\ x_i & \text{if firm } i \text{ is } \textbf{constrained}. \end{cases}$$

Upper-level problem: Nonsmooth nonconvex (G)NEP

The upper-level problem of firm i is then given by

$$\max_{x_i} \quad \sum_{t=1}^{T} w_t \cdot \left(\hat{P}_t(x) \cdot \hat{y}_{t,i}(x) - c_i \cdot \hat{y}_{t,i}(x) \right) - S_i(x_i, x_{-i}) \cdot x_i \quad \text{s.t.} \quad x_i \geq 0.$$

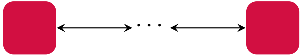
Challenges:

- The lower-level production gains can have kinks.
- The lower-level production gains can be nonconvex.
- Technical capacities at input nodes can result in shaRot1 constraints.

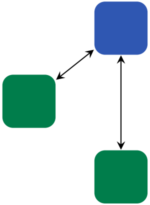
Observation: The hierarchical bilevel structure introduces new challenges. ($\longrightarrow$ Martin, Lars)

Recap: Systems of coupled optimization problems

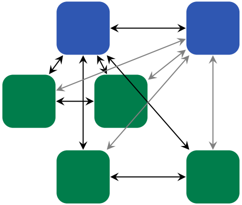
optimization problem



Nash equilibrium problem



bilevel problem



multi-leader-follower problem

Literature

- The primer on Nash equilibrium problems is based on the book **Spieltheorie: Theorie und Verfahren zur Lösung von Nash- und verallgemeinerten Nash-Gleichgewichtsproblemen** by C. Kanzow, A.S. (Birkhäuser Verlag).
  For an English and slightly extended version, contact me at alexandra.schwartz@tu-dresden.de.

- The multi-leader multi-follower example is based on the article **A tractable multi-leader multi-follower peak-load-pricing model with strategic interaction** by V. Grimm, D. Nowak, L. Schewe, M. Schmidt, A.S., G. Zöttl (MathProg 2021).